A Model of Balance-of-Payments Crises

INTRODUCTION

A GOVERNMENT CAN PEG the exchange value of its currency in a variety of ways. In a country with highly developed financial markets it can use open-market operations, intervention in the forward exchange market, and direct operations in foreign assets to defend an exchange parity (see [2] for an analysis of central bank operations and their effects on the exchange rate); the list could be extended to include such other instruments as changes in bank reserve requirements. But all of these policy instruments are subject to limits. A government attempting to keep its currency from depreciating may find its foreign reserves exhausted and its borrowing approaching a limit. A government attempting to prevent its currency from appreciating may find the cost in domestic inflation unacceptable. When the government is no longer able to defend a fixed parity because of the constraints on its actions, there is a "crisis" in the balance of payments.

This paper is concerned with the analysis of such crises. Although balance-of-payments crises have not received much theoretical attention, there are obviously features common to many crises, and the empirical regularities suggest that a common process must be at work. A "standard" crisis occurs in something like the following manner. A country will have a pegged exchange rate; for simplicity, assume that pegging is done solely through direct intervention in the foreign exchange market. At that exchange rate the government's reserves gradually decline. Then at some point, generally well before the gradual depletion of reserves would have exhausted them, there is a sudden speculative attack that rapidly...
eliminates the last of the reserves. The government then becomes unable to defend the exchange rate any longer.

It sometimes happens, however, that the government is able to weather the crisis by calling on some kind of secondary reserve: it draws on its gold tranche or negotiates an emergency loan. At this point there is a dramatic reversal—the capital that has just flowed out returns, and the government’s reserves recover. The reprieve may only be temporary, though. Another crisis may occur, which will obligie the government to call on still further reserves. There may be a whole sequence of temporary speculative attacks and recoveries of confidence before the attempt to maintain the exchange rate is finally abandoned.

One might question whether dramatic events of this sort, depending so heavily on the psychology of speculators, can be captured by a formal model. An analogy with another area of economics suggests, however, that sudden crises in the balance of payments may not be so hard to model after all. In the theory of exhaustible resources it has been shown that schemes in which the government uses a stockpile of an exhaustible resource to stabilize its price—an obvious parallel to using foreign reserves to peg an exchange rate—eventually end in a speculative attack in which private investors suddenly acquire the entire remaining government stock. The increase in private stocks is justified, ex post, by the increased yield on holding stocks; for when the price stabilization policy breaks down, the price of the resource begins rising, providing a capital gain that makes the holding of stocks more attractive.

In this paper I will show that a similar argument can be used to explain balance-of-payments crises. A speculative attack on a government’s reserves can be viewed as a process by which investors change the composition of their portfolios, reducing the proportion of domestic currency and raising the proportion of foreign currency. This change in composition is then justified by a change in relative yields, for when the government is no longer able to defend the exchange rate the currency begins depreciating.

Perhaps more surprising is that the pattern of alternating speculative attacks and revivals of confidence is also a natural event when the market is uncertain about how much of its potential reserves the government is willing to use. The reason is that speculators are faced with a “one-way option”; they do not lose by speculating against the currency even if fears of abandonment of fixed rates prove unjustified.

This paper, then, develops a theory of crises in the balance of payments. It is organized in six sections. Section 1 develops the macroeconomic model within which the analysis is conducted: a simple one-good, two-asset model originally expounded by Kouri [3]. In sections 2 and 3 the working of the model, and the evolution of the economy over time, are analyzed for flexible and fixed exchange rates respectively. Section 4 contains the central analysis of the paper, an analysis of the circumstances under which government pegging of the exchange rate suddenly collapses. This basic analysis is extended in section 5 to the case when government

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1This result was brought to my attention by Stephen Salant. A brief discussion of speculative attacks on government resource stocks is contained in [4].
policy is uncertain, producing the possibility of alternating crises and recoveries of confidence. Finally, section 6 discusses the significance and limitations of the analysis.

1. A MACROECONOMIC MODEL

In order to study balance-of-payments crises we must have a model with two characteristics: (1) the demand for domestic currency depends on the exchange rate; (2) the exchange rate that clears the domestic money market changes over time. An elegant and tractable model with these characteristics was developed by Kouri [3], and I will use a slightly modified version of his model to provide the underpinnings for the discussion. The model involves many special assumptions, and no claims are made for its realism. But it should become clear later that the main points of the analysis would go through in a variety of models.

We will assume, then, that we are dealing with a small country producing a single composite tradable good. The price of the good will be set on world markets, so that purchasing power parity will hold. That is to say,

$$P = sP^*,$$

where $P$ is the domestic price level, $s$ is the exchange rate of domestic currency for foreign, and $P^*$ is the foreign price level. I will assume $P^*$ fixed, so we can choose units to set $P^* = 1$. We can then identify the exchange rate with the price level.

The economy will be assumed to have fully flexible prices and wages, assuring that output is always at its full employment level $Y$. The balance of trade, which will also turn out in the model to be the balance of payments on current account, will be determined by the difference between output and spending:

$$B = Y - G - C(Y - T, W) \quad C_1, C_2 > 0,$$

where $B$ is the real trade balance, $G$ is real government spending, $T$ is real taxation, and $W$ is real private wealth (to be defined).

Turning now to the asset markets, investors are assumed to have available a choice between only two assets: domestic and foreign money. Both currencies bear zero nominal interest. The total real wealth of domestic residents is the sum of the real value of their holdings of domestic money $M$ and their holdings of foreign money $F$:

$$W = M/P + F.$$

As a final simplifying assumption we suppose that foreigners do not hold

\footnote{The reason for making this assumption is that it rules out international interest payments, allowing us to identify the current account with the trade balance.}
domestic money. Then $M$ is also the outstanding stock of domestic money and in equilibrium domestic residents must be just willing to hold that stock. Since I assume that the desired holdings of domestic money are proportional to wealth, the condition for portfolio equilibrium is

$$M/P = L(\pi) \cdot W \quad L_1 < 0,$$  \hspace{1cm} (4)$$

where $\pi$ is the expected rate of inflation. In this model $\pi$ is also the expected rate of depreciation of the currency.

The determination of $\pi$ is of crucial importance for the analysis, but it can more usefully be discussed in the context of a full dynamic analysis. For the moment I will treat $\pi$ as exogenous.

In this paper two exchange rate regimes will be considered. First will be a freely floating exchange rate, with the government abstaining from either buying or selling foreign money. Second will be a fixed exchange rate: the government holds a reserve of foreign money and stands ready to exchange foreign for domestic money at a fixed price. The short-run behavior of the economy under the two systems can be analyzed using Figure 1, in which the upward-sloping schedule $LL$ represents the condition for portfolio balance (4); an increase in holdings of foreign money will be accompanied by an increase in real domestic money for a given $\pi$. The downward-sloping schedule $WW$ represents the wealth constraint (3). To acquire foreign money at any instant, domestic residents must reduce their real holdings of domestic money.\(^3\)

Under a flexible rate regime, since neither the government nor foreigners will trade domestic money for foreign, there is no way for domestic residents to alter the composition of their aggregate portfolio. If they attempt to alter portfolio composition the effect will be to change the price level (exchange rate) instead. Suppose, for instance, that $\pi$ rises. This will make domestic money less attractive,
lowering $LL$ to $L'L'$. Since $F$ cannot change, $P$ rises, moving the equilibrium from $A$ to $B$.

Matters are different if the government has a reserve of foreign money $R$, and stands ready to exchange foreign for domestic money at a fixed price. Domestic residents can now trade freely up and down their wealth constraint, $WW$. An increase in $\pi$ that leads to a downward shift in $LL$ to $L'L'$ now leads to a shift in the portfolio of domestic residents, with the equilibrium moving from $A$ to $C$. There is a compensating change in the government’s reserve position as the government supplies the desired foreign money; the changes in asset holdings are related by

$$\Delta R = -\Delta F = \Delta M/P.$$ 

Thus, under flexible rates, changes in expectations are reflected in the short run in changes in the exchange rate; whereas under fixed rates they are reflected in changes in the government’s reserves. The next step is to examine the determination of expectations; this must be done in the context of an analysis of the economy’s dynamics.

2. DYNAMIC BEHAVIOR WITH A FLEXIBLE EXCHANGE RATE

If the government does not peg the exchange rate, the exchange rate can change for any of three reasons: a change in the quantity of domestic money outstanding, a change in private holdings of foreign assets, or a change in the expected rate of inflation. We will analyze each of these in turn, then combine them to describe the evolution of the economy over time.

I will assume that creation of money is dictated by the needs of government finance. Money will be created only through the government deficit; conversely, the government deficit will be financed entirely by printing money. Then the growth of money stock will be determined by

$$\dot{M}/P = G - T. \quad (5)$$

A convenient, if somewhat artificial, assumption is that the government adjusts its expenditure so as to keep the deficit a constant fraction of the money supply. If we let $M/P = m$, this means that $G$ is adjusted to make $G - T = gm$, where $g$ is constant. This in turn makes the rate of change of real balances depend only on the rate of inflation, for

$$\dot{m} = \dot{M}/P - (M/P)(\dot{P}/P)$$

$$= (g - \dot{P}/P)m. \quad (6)$$

Turning next to holdings of foreign money, recall that such holdings represent claims on the rest of the world. They can only be increased by exchanging goods in
return. So the rate of accumulation of foreign money must equal the current account balance.

\[ \dot{F} = B = Y - G - C(Y - T, W). \]  

(7)

Finally, we arrive at the question of expectations of inflation. This is a subject of considerable dispute. For the purposes of this paper it is essential to recognize that speculators are actively attempting to forecast the future in a sophisticated manner. This sort of sophisticated forward-looking behavior is best captured by the assumption of perfect foresight.⁴

\[ \pi = \dot{P}/P. \]  

(8)

To analyze the system as a whole, we begin by eliminating \( \dot{P}/P \). Recall the portfolio balance condition (4). Combined with perfect foresight, this function implies a relationship between real balances, foreign money holdings, and inflation, of the form

\[ \frac{\dot{P}}{P} = \pi(m/F), \quad \pi_1 < 0. \]  

(9)

The partial derivative in (9) follows from the fact that domestic residents will only be willing to increase the proportion of domestic money in their portfolio if they are offered a higher yield in the form of reduced inflation.

Substituting back, we get a dynamic system in the state variables \( m, F \):

\[ \dot{m} = [g - \pi (m/F)]m \]

\[ \dot{F} = Y - G - C(Y - T, m + F). \]  

(10)

This system is shown in Figure 2, with arrows indicating representative paths.

There are two points that should be noted about the dynamic system. First, even if we know the asset holdings of domestic residents, the exchange rate is indeterminate. For any arbitrary initial price level, given \( M \) and \( F \), we have an initial position \((m, F)\) and an implied path for the economy. The second point is that the system exhibits knife-edge instability. There is only one path converging to a steady state: if the initial exchange rate is not chosen so as to put the system on that path, the system will diverge ever further from the steady state.

A natural solution to both these difficulties is to assume that investors do not believe in the possibility of endless speculative bubbles, and that the initial exchange rate must therefore be one that implies eventual convergence to the steady state. Some theoretical justification for this assumption has been given by Brock [1];

⁴A more general assumption would be “rational expectations,” allowing for the existence of uncertainty. The special case of perfect foresight is easier to work with, however, and sufficient for present purposes.
the best argument for the assumption, however, is that it gives economically sensible results.

In Figure 2, then, the economy is assumed to always be on the stable arm $SABS$. If the initial holdings of foreign money are $F_0$, the price level will adjust so as to make the real domestic money supply be $m_0$, with the initial position of the economy being at point $B$. The system then converges gradually to $A$.

Notice that the real money supply depends positively on the stock of foreign money and is independent of the nominal stock of domestic money. Other things equal, then, the price level is proportional to the money supply and negatively related to $F$. We can write

$$P = MG(F) \quad G_1 < 0,$$

where (11) is the equation of the stable path $SABS$.

3. DYNAMIC BEHAVIOR WITH A FIXED EXCHANGE RATE

Suppose, now, that the government possesses a stock of foreign money $R$ and uses it to stabilize the exchange rate. This is, of course, equivalent to stabilizing the price level at some level $\bar{P}$. How does the economy evolve over time?

The easiest way to proceed is by examining the budget constraints of the private sector and the government in turn. The private sector can acquire assets only by spending less than its income. Let us define private savings as the excess of private income over spending,


Then from the budget constraint and the fact that the price level is pegged we immediately know that

$$\dot{W} = \dot{M}/\bar{P} + \dot{F} = S.$$
But private savings is in turn a function of private wealth, with \( \frac{\partial S}{\partial W} = -C_2 < 0 \). So (13) is a differential equation in \( W \), and since \( \frac{\partial S}{\partial W} \) is negative it is stable.

How is saving allocated between domestic and foreign money? This is determined by the portfolio balance condition (4). As long as investors believe that the government will continue to peg the price level, \( \pi \) will be zero and there will be a stable relationship between wealth and money holdings. Of a change in wealth, a proportion \( L \) will be allocated to domestic money and \( 1 - L \) to foreign money, so we have

\[
\frac{\dot{M}}{P} = LS
\]

\[
\dot{S} = (1 - L)S.
\]

(14)

The government can pay for its deficit \( G - T \) either by issuing new domestic money or by drawing on its reserves of foreign money \( R \). The government budget constraint can then be written

\[
\frac{\dot{M}}{P} + \dot{R} = G - T = g \cdot (M/P).
\]

(15)

As long as the government is committed to pegging the exchange rate, it has no control over how its deficit is financed. If the government issues more domestic money than the private sector is willing to hold, private investors can always withdraw the excess money from circulation by trading it for foreign money at the exchange window. As a result, the extent to which the government finances its deficit by running down its foreign currency reserves is determined by the private sector’s willingness to acquire additional domestic money:

\[
\dot{R} = -(G - T) + LS.
\]

(16)

An interesting point to note is that the rate of reserve loss does not stand in any one-to-one relationship with the trade balance. It can easily be shown that (16) implies the relationship

\[
\dot{R} = LB - (1 - L)(G - T),
\]

(17)

which can be either greater or less than \( B \).

Over time, then, both private wealth and government reserves will change. I illustrate the behavior of the two stocks in Figure 3. When the government runs a deficit it will lose reserves even if private saving is zero. As the paths illustrated by arrows show, pegging the rate ultimately becomes impossible if the budget is in deficit, no matter how large the initial reserves. If the budget were balanced, the lines \( \dot{R} = 0 \) and \( \dot{W} = 0 \) would coincide, and it would be possible for the economy to reach an equilibrium at the given exchange rate if initial reserves were large enough.

If the economy reaches an equilibrium with some reserves left, the model
developed above is just a particular case of the price-specie flow mechanism. When it is not possible to peg the exchange rate forever, the pegging effort will at some point collapse in a sudden balance-of-payments crisis. In the next section I analyze the nature and timing of such crises.

4. THE ANATOMY OF A BALANCE-OF-PAYMENTS CRISIS

In the last section I examined the behavior of an economy with a balance-of-payments “problem”; that is, of an economy gradually losing reserves. There comes a point when the problem becomes a “crisis”: speculators, anticipating an abandonment of the fixed exchange rate, seek to acquire the government’s reserves of foreign money. This crisis always comes before the government would have run out of reserves in the absence of speculation.

To see why this must be so, consider what would happen if investors did not anticipate the end of pegging. As long as the government has reserves left, the domestic money supply will be determined by the portfolio preferences of domestic residents \( M/P = L(\pi)W \), where \( \pi = 0 \). At the instant at which reserves are exhausted, portfolio balance begins to determine the price level instead of the money supply. The price level will immediately begin rising, for either or both of two reasons. Domestic residents may still be dissaving, and will try to reduce their holdings of domestic as well as foreign money; and, if the government is running a deficit, the nominal money supply must begin rising.

But when the price level begins rising, this will be reflected immediately in \( \pi \), by the assumption of perfect foresight. When \( \pi \) increases, the demand for domestic money falls and the price level jumps instantly by a discrete amount. The way this would happen is shown in Figure 4, which superimposes on the dynamic system of Figure 2 the position of the economy under fixed exchange rates. The ray \( OX \) is the expansion path of portfolios under fixed rates as private wealth changes; it is steeper than \( m = 0 \) because a higher proportion of domestic money is held in the portfolio when \( \pi = 0 \) than when \( \pi = g \) (as it does along \( m = 0 \)). When reserves run out the
system is at a point such as $A$. We know that when pegging ends and the exchange rate is allowed to float, real balances jump so as to put the system on the stable path $SS$. So the economy moves suddenly from $A$ to $B$. Because the nominal money supply is fixed at any instant, this occurs through a jump in the price level.

The argument I have just made depends on the assumption that when reserves run out the economy’s position is to the right of the intersection of the expansion path $OX$ with the stable path $SS$. Otherwise, the exchange rate would fall instead of rising when reserves run out. But it is easily shown that at the moment of exhaustion of reserves private wealth must be large enough to put the economy in the assumed position.$^5$ So if there is no speculation against the currency, the exhaustion of reserves will always produce a discrete jump in the price level, causing a windfall capital loss.

But investors cannot have expected such a capital loss to happen, because they would have avoided it. In particular, by exchanging domestic for foreign money an instant before reserves are exhausted, a speculator could earn an infinite rate of return. If everyone tried to do this, of course, the government’s reserves would be eliminated; the prospect of this would cause speculators to attempt to get out of domestic money still earlier, and so on.

The upshot of all this is that if investors correctly anticipate events, the reserves of the government must be eliminated by a speculative attack that enables all

$$m = G - T = L(0)S + L_1m \cdot \hat{\pi}.$$  

But $\hat{\pi} > 0$, because the share of domestic money in wealth is falling. So

$$L(0)S - (G - T) > 0.$$  

But under fixed rates, $\frac{\Delta R}{\Delta t} = L(0)S - (G - T)$. So, $\frac{\Delta R}{\Delta t} > 0$ at the intersection of $OX$ with $SS$.
investors to avoid windfall capital losses. Consider what such a speculative attack involves. From the government’s point of view, it represents a liquidation of its reserves. From the point of view of domestic residents, however, what they are doing is altering the composition of their portfolio, exchanging domestic for foreign money. If we let $M,F$ be the asset holdings of domestic residents just before the attack, and $M',F'$ be holdings afterwards, we know that

$$M'/\bar{P} = M/\bar{P} - R$$

$$F' = F + R. \quad (18)$$

Immediately following the attack, the economy is on a flexible rate regime. As discussed in section 2, the immediate post-crisis price level $P'$ can be determined from asset holdings:

$$P' = M' G(F') \quad (19)$$

or

$$P'/\bar{P} = (M'/\bar{P}) G(F') = (M/\bar{P} - R) G(F + R).$$

In order that there be no windfall capital loss, the speculative attack must not lead to a discrete change in the price level—that is, we must have $P' = \bar{P}$ or $P'/\bar{P} = 1$. It is this condition that determines when a balance-of-payments crisis occurs. For both $M/\bar{P}$ and $F$ are, under a fixed rate, functions of private wealth $W$. So the condition $P'/\bar{P} = 1$ can be written as an implicit function in $R$ and $W$,

$$1 = [L(0)W - R] G[W - L(0)W + R]. \quad (20)$$

Equation (20) defines a threshold in $W,R$ space. Under a pegged exchange rate $W$ and $R$ gradually evolve over time until they cross the threshold; then there is a sudden balance-of-payments crisis, which eliminates the remaining reserves and forces a transition to a floating exchange rate.

Figure 5 shows what happens in the crisis. Just before the speculative attack the economy is on the fixed-rate expansion path $OX$; just after, it is on the flexible-rate stable path $SS$. Suppose that, at the moment of the attack, private asset holdings are represented by point $A$. In the attack investors reallocate their portfolio, moving southeast along the line of constant wealth $WW$ to point $B$. The increase in holdings of foreign money is achieved by acquiring the government’s reserves $R$.

Suppose that, at the time of the crisis, private wealth had been larger—i.e., $WW$ had been further to the right. It is then obvious from the diagram that the reserves acquired from the government must also have been larger. This establishes that the threshold at which a crisis occurs is upward sloping in $W,R$ space.
The approach to the crisis is illustrated in Figure 6, where the threshold \((20)\) is represented by \(TT\); it is upward-sloping and cuts the horizontal axis to the left of \(R = 0\).\(^6\) We can learn something about the factors determining the timing of a crisis by comparing some representative paths like those leading from \(A\), \(B\), \(C\), and \(D\). \(B\) differs from \(A\), and \(D\) from \(C\), only in there being a higher initial level of reserves. In each case we can see that when reserves are larger, the absolute value of the change in private wealth before the crisis is larger. Since \(W\) is independent of \(R\), this means that the time until the crisis is longer. Thus we confirm the intuitively plausible result that the length of time for which a government can peg the exchange rate is an increasing function of its initial reserves.

When the government policy is certain, then, an economy with a balance-of-payments problem will pass through three stages: a period of gradually declining reserves, a sudden speculative attack, and a post-crisis period during which the currency gradually depreciates. The next step is to examine what happens if government policy is uncertain.

\(^6\)The intersection of \(TT\) with the horizontal axis corresponds to the intersection of \(OX\) with \(SS\) in Figure 5. But as argued in note 5, that intersection takes place at a level of wealth for which \(\dot{R} > 0\).
5. SPECULATION WHEN GOVERNMENT POLICY IS UNCERTAIN: THE "ONE-WAY OPTION"

Many different kinds of uncertainty could be introduced into the preceding analysis. I will deal with only one source of uncertainty: incomplete knowledge on the part of investors about how much of its reserves the government is willing to use to defend the exchange rate. This produces the possibility of alternating balance-of-payments crises and recoveries of confidence.

To consider the simplest case, suppose that the government’s reserves can be divided into a primary reserve $R_1$, which investors know it will commit to the defense of the exchange rate, and a secondary reserve $R_2$, which it may or may not be willing to use. We may suppose that the market believes that $R_2$ will be used with probability $\alpha < 1$. I also assume that once the government has used any part of $R_2$ to defend the exchange rate, the market can be sure that it will use all of it.

As before, we suppose that there is an initial period during which reserves gradually decline. Eventually there comes a point at which a speculative attack would take place if $R_1$ were the only reserve; but at that point there would not yet be a crisis if the market knew that the reserves committed to defending the exchange rate were $R_1 + R_2$. What happens?

The answer is that the speculative attack takes place, as investors acquire the whole of the government’s remaining primary reserve $R_1$. If the government then commits its secondary reserve to maintain the value of the currency, investors reverse themselves and exchange foreign for domestic money, producing a recovery of the government’s reserves.

To see why this must be so, consider two points. First, in the absence of transaction costs the speculative attack is costless. Investors need only hold a higher proportion of foreign money for an infinitesimally short period until it becomes clear whether or not the secondary reserve will be used. Second, if the capital outflow did not take place, there would be a possibility of a windfall capital loss. Suppose there were no speculative attack, or the attack was not large enough to completely eliminate the primary reserve. Then if the government eventually decided not to commit the secondary reserve, when $R_1$ was exhausted there would be a discrete jump in the exchange rate—a capital loss that an individual wealth owner could have costlessly avoided. So there must be a speculative attack just as if there were no secondary reserve. Once the secondary reserve is committed, of course, the risk of capital loss has been eliminated and the holdings of domestic money return to their previous level.

We can obviously extend this analysis to a whole series of reserves: $R_1, \ldots, R_n$. The effect is to produce a series of balance-of-payments crises, each ended by the government’s decision to commit the next reserve.

SUMMARY AND CONCLUSIONS

This paper has been concerned with the circumstances in which a balance-of-payments problem—defined as a situation in which a country is gradually losing
reserves—becomes a balance-of-payments crisis, in which speculators attack the currency. I have shown that balance-of-payments crises are a natural outcome of maximizing behavior by investors. When the government’s willingness to use reserves to defend the exchange rate is uncertain, there can be a series of crises in which capital flows out of the country, then returns, before the issue is finally resolved.

The analysis is subject to two major limitations. The first is that it is based on a highly simplified macroeconomic model. This makes it easier to develop the main points of the argument, but means that the analysis of the factors triggering a balance-of-payments crisis is incomplete. The second limitation is that the assumption that only two assets are available places an unrealistic constraint on the possible actions of the government, because the only way it can peg exchange rate is by selling its reserves. In a more realistic model we would have to allow for the possibility of other policies to stabilize the exchange rate, such as an open-market sales of securities or intervention in the forward market.

In spite of these limitations, however, the analysis is suggestive, and does help explain why efforts to defend fixed exchange rates so often lead to crises.

APPENDIX: THE DETERMINATION OF THE PRICE LEVEL UNDER FLEXIBLE RATES

In section 2 I derived a relationship between asset stocks and the price level under flexible rates from the requirement that the economy be on the stable path in Figure 2. An alternative algebraic derivation is the following. The dynamic system (10), linearized around the steady-state values \( \bar{m}, \bar{F} \), can be written

\[
\begin{bmatrix}
\dot{m} \\
\dot{F}
\end{bmatrix} = \begin{bmatrix}
-\pi_1 \bar{m}/\bar{F} & \pi_1 (\bar{m}^2 F) \\
-C_2 & -C_2
\end{bmatrix} \begin{bmatrix}
m - \bar{m} \\
F - \bar{F}
\end{bmatrix}.
\]

This system has the characteristic values

\[
\lambda_1 = -\frac{1}{2} (C_2 + \pi_1 \bar{m}/\bar{F}) - \frac{1}{2} \sqrt{(C_2 + \pi_1 \bar{m}/\bar{F})^2 - 4C_2 \pi_1 (\bar{m}/\bar{F})^2} < 0
\]

\[
\lambda_2 = -\frac{1}{2} (C_2 + \pi_1 \bar{m}/\bar{F}) + \frac{1}{2} \sqrt{(C_2 + \pi_2 \bar{m}/\bar{F})^2 - 4C_2 \pi_1 (\bar{m}/\bar{F})^2} > 0.
\]

A solution must be of the form

\[
\begin{bmatrix}
m - \bar{m} \\
F - \bar{F}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\
e^{\lambda_2 t}
\end{bmatrix}.
\]

If the system is to converge to a steady state the initial condition must be such that \( a_{12} = a_{22} = 0 \), so \( m \) and \( F \) converge exponentially to \( \bar{m}, \bar{F} \). But then we have
\[ \dot{m} = \lambda_1 (m - \bar{m}) \]

\[ = \pi_1 (\bar{m}/\bar{F})(m - \bar{m}) + \pi_1 (\bar{m}/\bar{F})^2 (F - \bar{F}), \]  

(A3)

which defines the stable path

\[ m - \bar{m} = \frac{\pi_1 (\bar{m}/\bar{F})^2}{\pi_1 + \pi_1 (\bar{m}/\bar{F})} (F - \bar{F}). \]

The rest of the argument in the text then follows.

LITERATURE CITED


