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THE INTENSITY AND SHAPE OF INEQUALITY: THE ABG METHOD OF DISTRIBUTIONAL ANALYSIS

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Inequality is anisotropic: its intensity varies by income level. We here develop a new tool, the isograph, to focus on local inequality and illustrate these variations. This method yields three coefficients which summarize the shape of inequality: a main coefficient, α, which measures inequality at the median; and two correction coefficients, β and γ, which pick up any differential curvature at the top and bottom of the distribution. The analysis of a set of 232 microdata samples from 41 different countries in the LIS datacenter archive allows us to provide a systematic overview of the properties of the ABG (α β γ) coefficients, which are compared to a set of standard indices including Atkinson indices, generalized entropy, Wolfson polarization, and the GB2 distribution. This method also provides a smoothing tool that reveals the differences in the shape of distributions (the strobiloid) and how these have changed over time.

JEL Codes: C16, C46, D31

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1. Introduction

The analysis of income distribution is central to our understanding of the structure of inequality and social transformations. In his seminal work on distributions, Pareto (1896, p. 99; 1897, pp. 305–24) proposed a leptokurtic distribution, which provided a good approximation to the top of the income hierarchy, and graphical representations based on incomes (Pareto, 1909, pp. 380–88). Improvements have been made since the introduction of the Gini index (Gini, 1914), but the outdated tools still in use have produced the general conception that inequality is a single-dimensioned concept, even though these tools can provide a variety of results.[1] The current contribution intends to show how local inequality can vary along the income scale. This idea is rooted in the traditional literature on the

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[1]There have been obvious improvements in our understanding of the socioeconomic processes which can generate these Pareto distributions (Gabaix, 2009), and even the double Pareto (Reed, 2001) since the lower tail has this particular shape as well. In this field, general surveys (Cowell, 1995; Kleiber and Kotz, 2003) illustrate the diversity of approaches. Over time, more appropriate and more general statistical distributions have been developed, from the Champernowne-I (Champernowne, 1937) and Fisk (1961) distributions to the Generalized B of the second kind (GB2), and are becoming standard tools (Jenkins, 2009). In addition, the graphical innovations used to represent distributions have been reviewed by Dombos (1982), who listed dozens of graphical models.

TABLE 1

Percentiles of Incomes in Israel and the U.S. in 2010 and the Difference between Them

	p1	p5	p10	p25	p50	p75	p90	p95	p99
il10	0.173	0.301	0.368	0.568	1.000	1.637	2.366	2.945	4.444
us10	0.057	0.235	0.362	0.611	1.000	1.531	2.171	2.731	4.501
Diff.	–0.116	–0.066	–0.006	0.043	0	–0.106	–0.195	–0.214	0.057

Note: Diff. shows the simple percentile level difference between the U.S. and Israel.

problem of ranking income distributions (Atkinson and Bourguignon, 1982; Shorrocks, 1983) and dominance issues (Yitzhaki, 1982), and is consistent with the development of inequality indices which are sensitive to specific segments of the distribution (Atkinson, 1970). Our aim here is to distinguish inequality at the middle, top, and bottom of the distribution.

Here, the measurement units are individuals and their income is the equivalized household disposable (after tax and transfers) cash income using the square root of the number of household members as an equivalence scale. This equivalized income is then divided by the median equivalent income of the population: "medianized equivalized disposable income" (medi) refers to this income concept. Zero or negative points have been excluded from the analyses. The same method can be adapted for the analysis of wealth inequality (Jäntti *et al*., 2013). This is a meaningful question for income distributions, as can be shown in an empirical example with datasets of the LIS database.[2] Table 1 shows the quantiles of the income distribution in Israel in 2010 (il10) and the U.S. in 2010 (us10). The Gini coefficients of both series are similar, at 0.387 and 0.371, respectively. However, the comparison of the distributions in Table 1 reveals considerable differences. In 2010 in Israel, the fifth percentile level (p5) was 30.1 percent of the median (p50), and percentile 95 (p95) was 2.95 times the median. Close to the median there was less inequality in the U.S. than in Israel. However, in the lower quantiles, the poorer Israeli residents were relatively better off than their U.S. counterparts by far, and the richest percentile p99 was closer to the median in Israel than in the U.S. Hence, in Israel, there was more inequality around the middle and less inequality at the extremes of the distribution, with this being particularly the case at the bottom. In terms of "general inequality" in 2010, as conventionally reflected in the Gini coefficient, for instance, Israel is slightly more unequal than the U.S. But in terms of "local" inequality, a notion that can be intuitively defined as a local stretching-out of the distribution, the Israel/U.S. comparison is obviously much more complicated, with there being both more and less inequality across various segments of the income distribution. This kind of ambiguous situation is related to the well-known problem of the comparison of Gini coefficients when the associated Lorenz curves cross each other. We aim to resolve this ambiguity by generalizing the idea of diversity in "local inequality"

[2]The runs on the Luxembourg Income Study (LIS) database had been completed in September 2014; we use the unambiguous two-character iso country codes.

over the income distribution.[3] We propose an analysis in terms of the shape of inequalities that has in general been neglected to date.[4]

We first discuss how the well-known Champernowne-I–Fisk (CF) distribution (Champernowne, 1937, 1953; Fisk, 1961) can be used as a baseline for local inequality analysis. From this baseline, we propose the "Isograph," a tool which represents the diversity of local inequality over the income distribution: this reveals how the empirical degree of inequality can be deducted from the CF hypothesis at the median but with additional curvature at the top and bottom of the distribution.[5] We therefore propose an α, β, γ(ABG) method of estimating three inequality parameters, compatible with the Pareto properties of the tails. The related coefficients are directly interpretable in terms of level-specific measures of inequality: the central coefficient (α) measures inequality at the median level, with correction parameters at the top (β) and bottom (γ). An empirical analysis of 232 datasets from 41 different countries provides estimates of the ABG coefficients. The ABG results are compared to 30 conventional indicators, and we also compare its ability to fit empirical distributions with that of the GB2, which can certainly be considered as the most influential distribution in contemporary income analysis (McDonald, 1984; Jenkins, 2009). The advantages of this ABG method are its ability to fit empirical cases, to help us understand the shapes of the distributions (strobiloids), and to provide interpretable coefficients.

2. The CF Distribution as a Baseline

The CF distribution is one of the many statistical laws used to model incomes. We cannot claim that the CF is the best curve—the GB2 provides a better fit since it is more flexible with two additional parameters—but it does provide a simple template which is able to pick up changes in local inequality.

In this CF tradition, we can approximate an income distribution as in equation (1). Consider each individual i ($i = 1, \ldots, n$) with income $y_i > 0$; she is above a proportion p_i of individuals (p_i is the so-called "standardized quantile" pertaining to income level y_i, otherwise called the "fractional rank") (see Jenkins and Van Kerm, 2009). The general quantile distribution expression of the CF of the shape parameter α (CF$_\alpha$) is particularly simple, provided that we consider medianized incomes (i.e., income divided by the median), $m_j = (y_j/\text{median})$:

(1)
$$\ln(m_j) = \alpha \ln(p_i/(1-p_i))$$
$$\text{or } M_i = \alpha X_i$$

where $X_i = \text{logit}(p_i) = \ln(p_i/(1 - p_i))$ is the logit-rank, and $M_i = \ln(m_i) = \ln(y_i/\text{median})$.

[3]Gabaix (2009) does consider this local degree of inequality, but his topics (mainly the size of cities, firms, and the largest actors on the stock market) lead to a focus on the top of the distribution and not on the whole scale.

[4]Weeden and Grusky (2012) recently focused on the forms of inequality, but in terms of categorical groupings rather than the distribution of economic resources.

[5]The isograph presents the slope of the "Fisk Graph" (Fisk, 1961, p. 176) that is indeed a logit-log transformation of the so called Pen's parade. In the Fisk Graph, compared to the early Fisk proposal of 1961, the axes are inversed (like in a quantile function) so that a log income pertains to a logit-rank position.

Expression (1) is precisely a CF_α, where α measures the degree of inequality understood as the stretching out of the distribution curve.

There are three types of strong arguments which support the use of a CF_α as a first approximation to income distributions.

First, with its two-parameter formula (the median and α), the CF is one of the most parsimonious laws with appropriate Pareto-type power-tails at both extremes, and its formula is remarkably simple. In the CF, log medianized income is proportional to the log-odds of the standardized quantile. This parsimony is notable, and the coefficient $\alpha \in]0, 1[$ in the CF_α has a remarkable role in the measurement of inequality since its value is the Gini coefficient.[6]

Second, the CF has a particular position in the field of distributions (McDonald and Xu, 1995, p. 139). It is central in the general tree of beta-type distributions (Kleiber and Kotz, 2003, p. 188) where GB2 is in this sense the canopy of the tree and the CF the roots. The CF is a very simplified GB2 where the parameters p and q equal 1. While the CF is much less flexible than the GB2, it does share some important features, such as power-tails. The CF is a sub-case of the complete Champernowne-II (Champernowne, 1937, 1952) four-parameter distribution; Fisk (1961) described this simplified form more generally. He called this the "sech^2 distribution" (the square of the hyperbolic sequant); it is also called the log-logistic distribution (Dagum, 1977, 2006; Shoukri *et al.*, 1988).

Third, the CF produces income distributions that are solidly grounded in mathematical expressions. Here the CF is at a crossroads of different theoretical traditions. In microeconomics, the GB2 (and, as a consequence, the CF which is a GB2 with parameters $p = q = 1$) can be seen as a result of Parker's neoclassic model of firm behavior (Parker, 1999, p. 199; Jenkins, 2009). A number of other theoretical constructions, such as stochastic processes of income attainment, yield the same distribution. In a proposal from the field of finance, Gabaix (2009) considers stochastic models based on geometric Brownian motion that can generate this type of distribution.

In the social sciences, the balance of power theory of incomes also generates CF laws. This theory assumes proportionality between the power of income and the power of rank. Developed societies are socially hierarchized on the basis of rank (of education, prestige, political power, or "value" of any kind) which can be expressed as a standardized rank p in $]0,1[$. Each individual i ($i = 1, \ldots, n$) with income y_i is above a proportion p_i of individuals and has a proportion of $q_i = 1 - p_i$ individuals above him. Since the "power of income" (Champernowne, 1937) is defined as $Y_i = \ln(y_i)$, the "power of social rank" (or "logit rank," or also "logit quantile") can be defined as the logit of the rank quantile p_i: $X_i = \ln(p_i/q_i) = \text{logit}(p_i)$.[7] Consider two individuals (i) and (j): their difference in power of income $\Delta Y = Y_j - Y_i$, is proportional to the difference in their power of rank, $\Delta X = X_j - X_i$. Then, $\Delta Y = \alpha \Delta X$, where

[6]With his parameterization of the CF cumulative-distribution function $F(k; \lambda; \delta) = (1 + \lambda k^{-\delta})^{-1}$, $k > 0$, $\lambda > 0$, $\delta > 1$, Dagum (2006, p. 245) demonstrates that Gini $= \delta^{-1}$, where $\delta > 1$ is the shape parameter of the Fisk distribution. In particular, the α of the ABG is equal to Dagum's δ^{-1}, so Gini $= \alpha$ provided that $\alpha < 1$. In the case of a discrete population, α can be greater than 1: an example is the distribution of the number of war casualties over the last century (Cederman, 2003), where α is estimated to be 1.5. With income distributions, the highest Gini coefficients are below 0.7.

[7]Among others, Clementi *et al.* (2012) log-transform the value of rank, even though the quantile, which is an $]0,1[$ interval variable, should be transformed symmetrically (around 0.5), which is what the logit transformation does, as in Copas (1999).

the constant α reflects the intensity of economic inequality in this society. The income inequality between (i) and (j) can thus be derived from the social power of rank:

$$\ln\left(y_j/y_i\right) = \alpha \ln\left(\left(p_j/\left(1-p_j\right)\right)\left(\left(1-p_i\right)/p_i\right)\right) \tag{2}$$

The higher is p_i, the greater is the power of social rank; as p_i tends to 1, the power of social rank tends to $+\infty$. This could explain why, at the top of the distribution of prestige, it is strategic to increase rank, as the rewards in terms of logit(quantile) tend to infinity, and the cost of losing rank is very high, and obviously much larger than that in the neighborhood of the median. Equally, close to the bottom, gaining/losing rank may have immense consequences in terms of the power of rank and relative income. This could explain why Aristotle sees the top of the distribution as dangerously arrogant and the bottom prone to brutality, while the middle of the scale corresponds to stability and moderated political attitudes (Aristotle, 1944, p. 329). One important consequence of equation (1) is the existence of a "sling effect," since, as α increases, the consequences of a percentage change in income can be significant close to the median but critical at the extremes of the distribution.

In detail, under a CF_α distribution, a change of one percentage point in α generates an increase of income of about one percentage point near the third quartile ($X = 0.098$), about two percentage points near the ninth decile ($X = 2.197$), about three near the top 95 percent ($X = 2.944$), and so on. As the Gini coefficient rises, extreme top-incomes gain a much higher percentage in terms of their initial income than do the upper middle class. Symmetrically (in terms of log), the poor suffer from greater percentage declines in resources than do the lower middle class.

A number of different fields of research (microeconomics, finance, statistics, and social sciences) thus confirm the importance of the CF, although the adequacy of its description of empirical reality remains to be established. The CF is not the best curve in general: since the GB2 has two additional parameters, it should provide a better fit. Even so, the CF is a parsimonious relevant baseline or template for inequality, empirically relevant in the field of income distribution.

3. Measuring Empirical Divergences from the CF Distribution

The analysis of empirical distributions confirms that expression (1) is a first-order approximation that can be improved upon (Appendix 1). I propose the introduction of an ISO function that generalizes (1) into equation (3) and, thereby picks up the divergence of the empirical curve from the CF hypothesis:

$$M_i = \mathrm{ISO}(X_i)X_i, \text{ where } M_i = \ln(y_i/\text{median}). \tag{3}$$

Simply, ISO represents the ratio M/X. If ISO(X_i) is a constant (α), (3) simplifies to (1) and the distribution is a CF_α that equals the Gini index; the higher the value of α, the greater is inequality.

In general, the CF distribution hypothesis does somewhat diverge from reality. Therefore, the isograph representing ISO(X_i) is not a constant and expresses the intensity and the shape of local inequality. The higher is ISO(X_i), the greater the stretching out of incomes at the logit rank level X_i. The change in ISO(X_i) along the distribution measures "local inequality," which can be thought of as the local stretching of the distribution.

The empirical isographs are horizontal lines that are often bent at the two extremes in different ways. These are obtained empirically by graphing the ISO(0) for each "vingtile" (the 19 slices of five percentiles from 5 to 95 percent). The value of ISO(0), which can be erratic, is replaced by the average of ISO($p = 0.45$) and ISO($p = 0.55$). The shape of these curves can be explained by taxes, social and redistributive policies which can distort the income curve in such a way that the ISO is not constant. The poor can either benefit from income support or be the victims of extreme social exclusion. The rich can either organize a system of resource hoarding or accept the development of massive redistributive policies. Therefore, the hypothesis of the strict stability of α along the income scale generally does not hold, since power relations can be stronger or smoother at the top and bottom of the social ladder than near the median.

When the isograph is relatively flat (for example, Finland in 2004), α equals the Gini index (0.24 for fi04 in Figure 1). In France, Germany, and Brazil, the CF distribution hypothesis is an acceptable first-order approximation, but in other countries the isograph is obviously not constant. The isograph more often reveals a declining level of inequality at the top of the distribution (an ISO with negative slope). An extreme case is Israel in 2010, with an ISO close to 0.50 at the middle of the distribution, similar to Brazil, but much lower at the ends. At the bottom 5 percent of the Israeli distribution, ISO(–3) = 0.40, which is similar to Spain and much less than the U.S. figure, and at the top 5 percent of the distribution the Israeli ISO(3) = 0.36, which is very similar to that in the U.S. These findings illustrate the large movements in local inequality over the income hierarchy. The

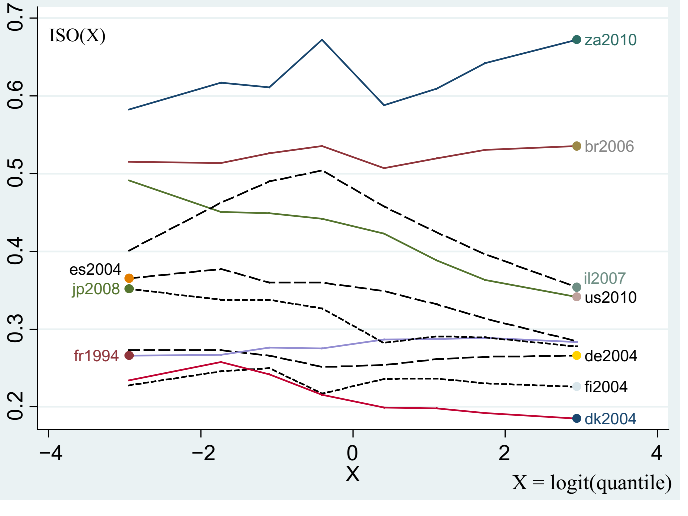

Figure 1. The Isograph in 10 Contrasting Cases

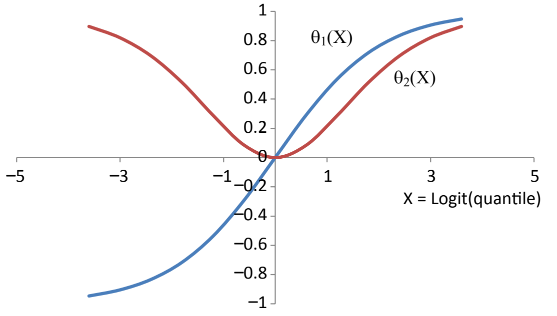

Figure 2. The θ_1 and θ_2 functions

crossings of the isographs for Israel and the U.S. show extreme inequality close to the median in Israel balanced by more equality at the extremes. The isograph helps us to compare inequalities that can shift over the income distribution.

4. The ABG Method of Parametric Estimation of the ISO

The shapes of the 232 isographs (see Appendix 1) show that they can be accurately captured by only three parameters that I introduce here.[8] The isograph shapes show that a coefficient pertaining to the level of local inequality close to the median (α) can be defined along with two shape parameters reflecting isograph curvature at the two extremes. Two correction coefficients β and γ are therefore determined, where $\alpha + \beta$ is the upper asymptote of the ISO and $\alpha + \gamma$ the lower asymptote. When γ and β are zero, the distribution is CF_α with coefficient which is the Gini. The added value of this method is to deliver unambiguous interpretable parameters of inequality showing both local inequality at the median, and corrections at the top and bottom of the income distribution.[9]

The parameterization proposed here is compatible with the well-established hypothesis that the upper tail has a power-tailed Pareto-type shape (Piketty, 2001), so that the upper asymptote of the ISO(X) function should be a zero-slope line of the equation ($Y = \alpha + \beta$). We hypothesize, following Reed (2001), that the lower tail is also Pareto-shaped. Thus, the lower asymptote of ISO(X) should be the zero-slope line of the equation ($Y = \alpha + \gamma$). Between these two, we have smooth changes.

The parametric expression for these curvatures is based on two functions θ_1 and θ_2 related to hyperbolic tangent functions: $\theta_1(X) = \tanh(X/2)$ and $\theta_2(X) = \tanh^2(X/2)$ (see Figure 2).

We use two simple linear combinations of these θ functions, B and G, to make the coefficients easier to interpret. Consider the adjustment of ISO defined by:

[8]Three plus one parameter of scale that disappears in the case of medianized incomes. GB2 and ABG have the same number of parameters, i.e., three of shape and one of scale.

[9]This aspect is important: the GB2 distribution proposes, in general, a good fit of empirical distributions (Jenkins, 2009), but the interpretability of its p and q shape coefficients is unclear.

TABLE 2

Estimates of ABG Parameters in Israel and the U.S. in 2010

IL2010	Coefficient	S.E.	95% C.I. min	95% C.I. max
α	0.53852	0.00059	0.53737	0.53968
β	–0.23972	0.00124	–0.24215	–0.23728
γ	–0.14505	0.00114	–0.14728	–0.14282
	N = 18,936	r^2 = 0.9959		
US2010	**Coefficient**	**S.E.**	**95% C.I. min**	**95% C.I. max**
α	0.42699	0.00005	0.42689	0.42709
β	–0.09251	0.00015	–0.09280	–0.09223
γ	0.05202	0.00031	0.05141	0.05263
	N = 191,055	r^2 = 0.9991		

$$\text{(4)} \qquad \mathrm{ISO}(X_i) = \alpha + \beta B(X_i) + \gamma G(X_i)$$

$$\text{where } B(X) = \frac{\theta_1(X) + \theta_2(X)}{2} \text{ and } G(X) = \frac{-\theta_1(X) + \theta_2(X)}{2}$$

$$\text{and } \theta_1(X) = \tanh(X/2) \text{ and } \theta_2(X) = \tanh^2(X/2)$$

then,

$$\text{(5)} \qquad M_i = \alpha X_i + \beta B(X_i) X_i + \gamma G(X_i) X_i$$

where $X_i = \mathrm{logit}(p_i)$ and $M_i = \ln(m_i)$.

Equation (5) is estimable as X_i and the functions are known, and there are no collinearity issues. The α, β, γ can be estimated in a single multivariate OLS regression without a constant. In the results:

- the coefficient α measures inequality close to the median;
- β characterizes the additional inequality at the top of the distribution, β being positive when the rich are richer than in the CF_α, so that the upper tail is stretched; and
- γ characterizes the additional inequality at the bottom of the distribution, with γ being positive when the poor are poorer than in the CF_α.

The first comparative example refers to the estimation of the ABG ($\alpha\ \beta\ \gamma$) coefficients on Israeli and U.S. data in 2010. In each sample, individuals are defined by their logit(quantile) of income and their related B and G functions. The OLS linear regression we propose is easy to carry out and produces the estimates of the ABG parameters and their standard errors in Table 2.[10] The results reveal that at the middle of the distribution, there is more inequality in Israel than in the U.S., but the negative coefficients on the curvature parameters β and γ show that there is less

[10]To control for the potential problem of outliers, the regression interval is reduced to abs(X) < 4, which means the two centiles at both extremes are excluded from the regression. A second cut-off estimation has an alternative span of abs(X) < 8. The results for α are not affected (the correlation between the two series is $r = 0.9998$), and those on β and γ remain unchanged ($r = 0.9886$ and $r = 0.9694$, respectively). This choice does not then particularly affect the results. Furthermore, in this case the r^2 are comparable, even though in the general case of regressions omitting the constant term, the ratio between the regression sum of squares and the total sum of squares makes no sense. Here is an exception since, for the observed and for the estimated series, at the median level, both logit(p_i) and ln(m_i) are null.

inequality in Israel at the extremes than in the U.S. ($\alpha + \beta$ and $\alpha + \gamma$ are both smaller in Israel). These results reflect the complex comparison of the U.S. to Israel in Table 1, and underline the particular polarization in Israel (García-Fernández *et al.*, 2013); conversely, in the U.S., there is extreme inequality at the bottom (high values of γ) with very low values for the poorest centiles.

In the empirical analysis of 232 cases, β and γ are always much smaller than α (see Appendix 2): in this case, the CF distribution is an acceptable simplified first-order hypothesis, and β and γ are correction coefficients. When β (γ) is 1 percent higher, the ISO(X) function increases by 1 percent at the upper (lower) asymptote. The ABG has three shape parameters, plus one scale parameter derived from the ISO(X) estimation function (6).[11]

In this decomposition, α, $\alpha + \beta$ and $\alpha + \gamma$ are the inequality measures at the median, top, and bottom of the distribution, respectively, and are homogeneous with the Gini coefficient in the sense that the upper tail of a distribution of coefficient ($\alpha + \beta$) is similar to a $CF_{\alpha+\beta}$.

There is no analytic expression for these measures since they come from a regression of M on the functions XB(X) and XG(X). Similarly, equation (5) yields no simple cumulative distribution function, but when $\beta = \gamma = 0$, equation (5) corresponds to a CF_{α} distribution. The solutions are numerical whenever β or γ is non-zero.

The coefficients α, β, and γ satisfy most of the criteria of the appropriate inequality measures (see Jenkins, 1991, 1995; Cowell and Jenkins, 1995; Haughton and Khandker, 2009, p. 105):

- Mean independence: a proportional change in incomes does not affect the measures.
- Population-size independence: all else equal, a change in population size does not affect the measures.
- Symmetry: if individuals (a) and (b) exchange their income levels, the measures are not affected.
- Decomposability: this possibility is not exploited in the limits of the current paper, but covariates can be added to model (5) so that nested models can show how inequality results from inter- or intra-group variance, with the "group" being potentially defined by gender, education, ethno-cultural origins, and so on.
- Pigou–Dalton Transfer (PDT) sensitivity: the ABG method and the idea of local inequalities is not compatible with the strict PDT principle which claims that inequality falls when a richer individual (a) gives a part of her income to a poorer individual (b), provided that the hierarchy is not inverted. If (a) and (b) are above the median, and if the local inequality between (a) and (b) falls, inequality between the median and (b) increases since (b) gets richer, and thus further from the median. Such a transfer is ambiguous at the local level: even if the stretching between (a) and (b) is lower, meaning less inequality, the stretching between (b) and the median increases, meaning more inequality. The ABG method does satisfy, in any

[11]In the conventional literature, this is a 4-parameter distribution, but with medianized income the traditional b coefficient is automatically set to 1.

case, a weaker form of the PDT principle provided that (a) is above the median and (b) below it, and that they remain in this order relative to the median after the transfer.

5. Comparative Analysis of 232 Datasets and Inequality Measures

Sections 5 and 6 analyze the performance of the ABG method compared to existing indicators (Section 5) and to the well-known GB2 distribution (Section 6). The added value of the ABG method over other measures is illustrated via its comparison to more customary inequality indices on a set of 232 harmonized microdata files covering 41 countries provided by the LIS datacenter.

The first result is that the absolute values of β and γ are small compared to that of α, so that $\alpha+\beta$ and $\alpha+\gamma$ are always in the interval [0,1]. The signs of β and γ can be positive or negative (Figure 3), and the point $\beta = 0$ and $\gamma = 0$ is in the middle of the range of βs and γs.

We can compare the three ABG indices to other standardized inequality measures. These selected indicators are well-known or based on simple income ratios. We consider ISO indicators at five different levels. In addition, the sizes (as a proportion in the total population) of five income classes are included: the poor (po), lower middle class (mcl), middle class (mc), upper middle class (mcu), and the rich (ri). Overall, our analysis covers a set of 30 variables and 232 data samples (see Appendix 2a).

One important question is the relative position of the ABG parameters in the field of inequality measures. A first answer is given by an analysis of the correlation matrix of these indicators (Appendix 3): there is a very strong relation between α

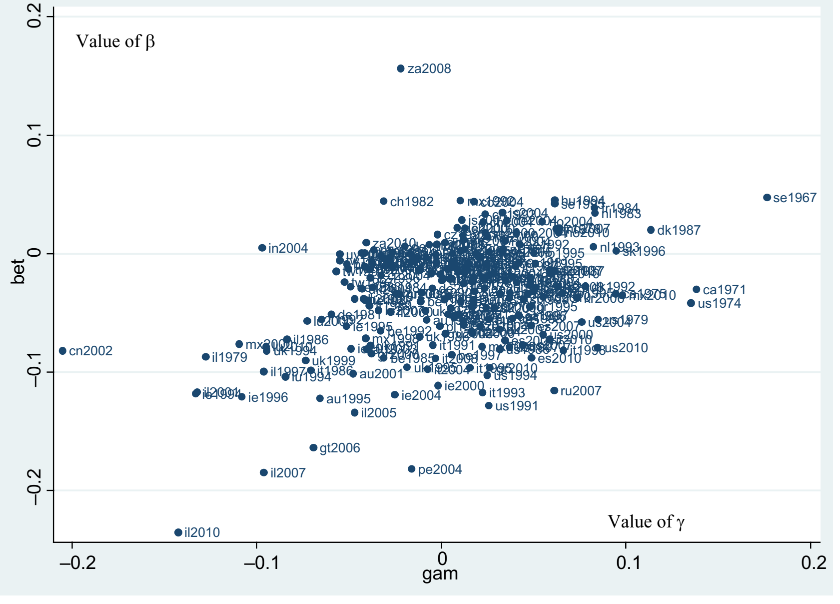

Figure 3. The Relation between β and γ

and the Gini index ($r = +0.95$), thus confirming the relation of these two inequality measures when the CF_{α} approximation is acceptable. More generally, most of the measures correlate well with α. This is good news for the ABG method, but then what is its intrinsic added value? A second answer is that we also see interesting correlations for the β and γ coefficients, which thus provide complementary information to α: the degree to which inequality moves at the top and at the bottom of the distribution. A third more systematic answer comes from the principal component analysis (PCA) of the whole table (see Appendix 2a). The first axis of the PCA (69 percent of the total variance) reveals the similar nature of many inequality measures, including α; this axis picks up inequality intensity. The α coefficient appears on the first axis of the PCA, along with the Atkinson index with parameters 1 and ½, the generalized entropy with parameters 1 and 0, the Gini coefficient, a number of quantile ratios, as well as the Wolfson polarization index.[12] This confirms that α is a new inequality parameter which is highly correlated with the main inequality measures, but is more sensitive (like the Wolfson index) to the median of the distribution (see Appendix 2a for details).

The role of β and γ becomes apparent on axes 2 and 3 (12 and 7 percent of the variance, respectively), which reveal the shape of inequality but not its intensity. On the second PCA axis, β and γ are strongly correlated in the same direction as the two measures of the over-elongation of the extreme deciles compared to the quartiles:

$$\text{pp251050} = (\text{quartile1}/\text{decile1})/(\text{median}/\text{quartile1}), \text{ and its symmetric,}$$
$$\text{pp907550} = (\text{decile9}/\text{quartile3})/(\text{quartile3}/\text{median}).$$

The correlation with the upper and lower middle class sizes is negative: the elongation at the top (resp. bottom) implies a smaller upper (resp. lower) middle class that is stretched out. Positive values on the second axis reflect greater inequality at the extremes. Here, the generalized entropy index with parameter 2 is more strongly correlated on axis 2 than are the other traditional measures.

Axis 3 shows the difference between γ and β, along with the contrast between pp251050 and pp907550. On this axis, the generalized entropy index with parameter 1 and the Atkinson index with parameter 2 are located on the same side of axis 3 as γ. All of these indicators are relatively more sensitive to inequality at the bottom. Conversely, the generalized entropy index with parameter 2, located on the same side of axis 3 as β, is sensitive to inequality at the top. Therefore, β and γ pick up salient features of the distribution that are less-well detected by other measures.

We can also ask whether the ABG method provides a better assessment of polarization than the Wolfson index (Wolfson, 1986). A simple measurement of middle class hollowing, rpol ratio, is the of the sum of the size of upper middle class and lower middle class by central middle class, the one closer to the median. The linear correlation matrix shows that, in terms of r^2, the Wolfson index is indeed better than the Gini in predicting the rpol ratio, but α is even better. A nested model of the 232 datasets with respect to middle-class polarization (rpol) compares the performances of the Gini, the Wolfson, and the α coefficients (see Appendix 2a). In general,

[12]The Wolfson index is chosen here because it is standard in the literature, even though more reliable propositions exist (Alderson *et al.*, 2005; Chakravarty and D'Ambrosio, 2010).

for the different aspects of inequality measurement, the ABG method offers interpretable parameters that generally outperform the other methods in terms of the description of the distribution and the size of income classes.

6. How do the ABG Distribution and GB2 Perform?

Another aspect of the ABG method is its distributional shape: the three parameters describe a distribution that is tailored to fit the observed data. How does ABG perform in this respect? In the contemporary income-distribution literature, the GB2 is the leading contender for the best measure (Jenkins, 2009; Graf and Nedyalkova, 2014). This distribution is particular in the universe of beta-type laws: it is the most general, as many other distributions are special cases. It has four parameters, one of scale (b) and three of shape (a, p, q), which is the same number as the ABG distribution, provided that we consider equation (5) as a general expression of an empirical distribution where the fourth parameter (size) is the median. In terms of microeconomic theory, the GB2 results from a simple model of firm behavior (Parker, 1999), and is acknowledged for its flexibility. Statistical tools to estimate the GB2 parameters are easily available.

To compare the respective performances of ABG and of GB2, we consider the divergence from the empirical observed distribution (OBS) of ABG, GB2, and CF. This is not an easy task since the GB2 predicted values are based on a known cumulative distribution function and an unknown quantile function (although its estimation via simulation is possible) and the ABG provides a quantile function. Our solution here is to compare the predicted values of each "vingtile" level of logged incomes for four quantile functions: the empirical distribution as the target, the GB2 and ABG as competitors, and the CF_α as the baseline. As they have more parameters (and are thus more flexible), the GB2 and ABG provide a better fit to the OBS than does CF_α. One measurement of the goodness of fit is the r_a^2 (the adjusted coefficient of determination): the higher is r_a^2, the better the fit to the OBS. The most difficult issue concerns the estimation of the GB2 parameters (a, b, p, q) for the 232 samples; here the STATA gb2lfit program only converged quickly in 205 cases. The maximum number of iterations was set to 6, since convergences after 7 or more iterations are exceptional and may be considered as outliers.

Our analysis is restricted to the 205 convergent cases. We have for each country the vectors of 19 vingtiles of log income levels for the CF-Gini (linc_q), ABG (lina_q), GB2 (ling_q), and the empirical OBS distribution (lino_q: o for observed), with $q = 1 \ldots 19$. The adjusted r_a^2 of the OLS regression of (linc_q) on (lino_q) reflects the quality of the CF hypothesis: the higher the value, the better the fit (Table 3). On average, the CF is a good first-order approximation $\left(r_a^2 = 0.996\right)$, and both GB2 and ABG improve the fit further, with a clear advantage for the latter. In 67.8 percent of cases, GB2 is better than CF, but ABG outperforms CF in 85.8 percent of cases and GB2 in 76.5 percent.

We can explain the better fit of the ABG methodology. We simulated many GB2 distributions from the shape parameters a, p, and q, each randomly-defined; we then fitted these with the ABG method, and found no cases where the β and γ

TABLE 3

Frequency of a Better Fit of Distribution D1 Compared to D2 (%) on 205 Samples (27 excluded cases with more than 6 iterations)

	Average Adjusted r^2 of the fit of OBS	Pair Comparison: % of Cases Where the Fit of A is Worse Than That of B
CF	0.9959	CF worse than GB2: 67.8%
GB2	0.9975	GB2 worse than ABG: 76.5%
ABG	0.9989	CF worse than ABG: 85.8%

coefficients were strongly negative at the same time. This means that strongly polarized distributions such as than in Israel in 2010, with its stretched middle class and relatively more equality at the top and the bottom, cannot be generated from the GB2 distribution. The GB2 is flexible, but does not cover every case, and in particular those where β and γ are both positive. This means that the GB2 with parameters a, p, q is less general than the ABG with coefficients α, β, and γ, which is more flexible with the same number of parameters.

The GB2 is a good tool, and has the advantage of being theoretically more solid and mathematically purer than the ABG, but does nonetheless present some difficulties. The interpretation of the GB2 parameters a, p, q is not obvious, with the exception of the case where $p = q = 1$. The ABG method is on the other hand less theoretically-satisfying: it has no simple analytical expression, is very empirical, and is a computer-oriented fitting tool. However, ABG produces three easy to estimate and interpret coefficients that make sense of the distribution of inequality, with values that are compatible with the Gini tradition since α = Gini if β and γ are close to 0.

7. Representing the Shape of the Income Distribution: The Strobiloid

The ABG decomposition provides a method for smoothing the empirical quantile distribution function. If, for instance, we are interested in the architecture of societies represented by the distribution density curve, as in the seminal work of Pareto (1897, p. 315), we can plot income on the vertical hierarchical axis and the density value on the horizontal axis, as in Figure 4. One convenient way of standardizing the representations, for comparison purposes, is to normalize the income curve. With both the medianization of income and the normalization of the surface to 1 (so that it defines the density of the distribution), we can superpose the curves for different periods or countries. This is the strobiloid representation (Chauvel, 1995; Lipietz, 1996; Chauvel, 2013).[13]

These empirical strobiloids reveal the diversity of income distributions across countries and reflect the change in socioeconomic architecture within countries. In the strobiloid, the wider the curve, the more individuals there are at this level of the graph: middle-class societies will have a large belly (Denmark), whereas in the contemporary American distribution a large proportion of the population is close

[13]The strobiloid is based on Pareto's (1897, p. 313) idea that the shape is one of an arrow or of a spinning top. This representation, which is similar to Pareto's first representations of the income pyramid, allows us to make 2 • 2 comparisons over countries, time, etc. Nielsen (2007) provides an overview of Pareto's legacy, and considers why this has generally been neglected in the social sciences.

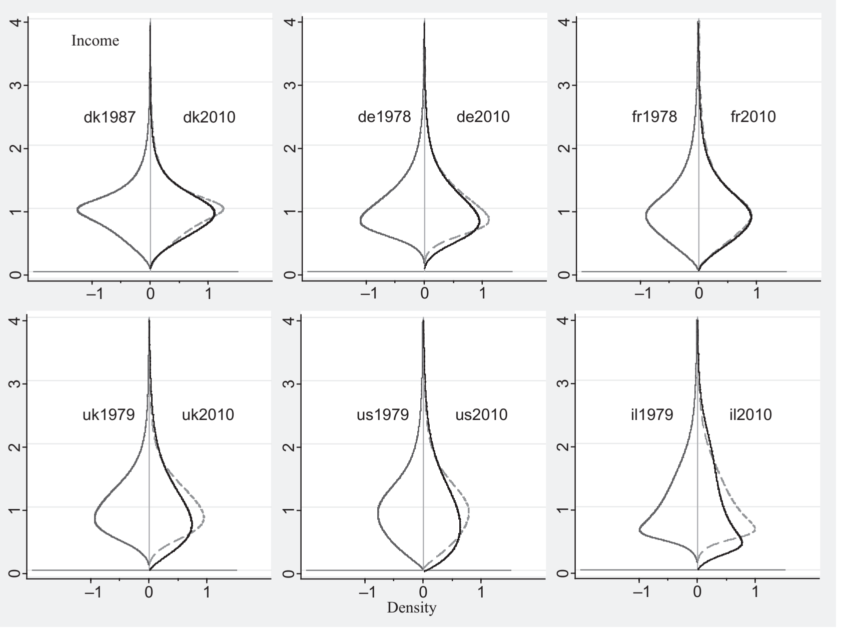

Figure 4. Six typical strobiloids (Denmark, Germany, France, U.K., U.S., and Israel)

Note: The strobiloid shows the income hierarchy (on the vertical axis, 1 = median). The curve is larger (horizontal axis) when the density at this level of income is higher: Many individuals are at the intermediate level near to the median and their number diminishes at the top and at the bottom. Thus, in strobiloids with a larger belly, the intermediate middle class is larger with a more equal distribution.

to the bottom. Kernel smoothing can produce similar curves, but the ABG method relies on a Pareto power-tail compatible methodology to produce interpretable parameters. This new tool allows the country and time comparison of the considerable developments in the intensity and shape of inequality.

The strobiloid shows that incomes in Denmark in 1987 are generally "more equal" than elsewhere, although the particularity of Denmark (due to its low α and high positive γ) is its lack of rich rather than its lack of poor, with some of the latter being stretched far to the bottom of the distribution. The bottom part of the curve in Germany in 1983 shows the same level of inequality as in Denmark in 1987 (as can be seen from the isograph in Figure 5), although there is more inequality in Germany in 1983 for higher income levels. In terms of public policy, the structure of Germany in 1983 is a particular model of homogeneity below the median with a high implicit level of minimum income.

The French distribution is fairly common in Europe and is stable over the period under consideration. On the contrary, there is a strong polarization trend in the U.K., which is converging to the onion-shaped strobiloid of the U.S. The U.S. itself has an even more pronounced onion shape with increasing inequality. One feature of this shape is less the extreme values at the top but rather the lower values with a very high γ. Israel, the final case, may be the most symbolic in terms of the shift from a more equal to a far less equal distribution, with one particular feature: a steady decline in

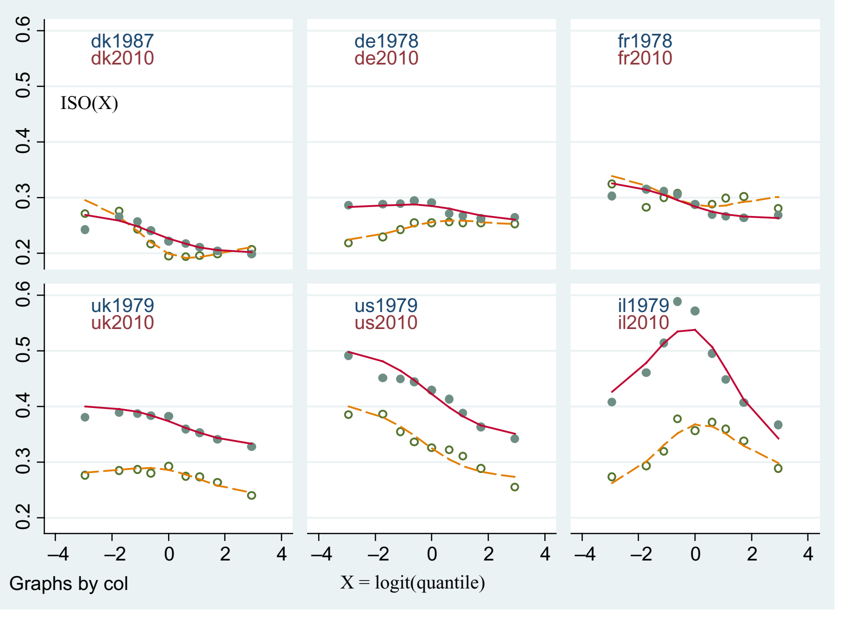

Figure 5. Isographs for Six Typical Countries

Note: The dots represent the empirical values and the lines are the fitted isographs (ABG method). For each country, two periods are considered: the dashed line and white dots pertain to the older years, the full line and gray dots refer to more recent years. The higher the curve at a given level of X (logit rank), the greater are the income inequalities at this level. Israel over 1986–2010 is an obvious case of extreme polarization.

the median class of incomes with a relatively strong minimum-income scheme, leading to the development of an unprecedented arrowhead-shaped curve. Israel appears then as an extreme case of rapid polarization over recent decades, which is confirmed by the isograph in Figure 5. A broader international comparison reveals the diversity of distributions across countries (Appendix 4).

8. Conclusion

The ABG methodology represents progress in terms of both measurement and graphical representation (CF curve, isograph and strobiloid) of the diversity of inequality at different levels of income, since in many cases inequality is anisotropic along the income scale. In terms of public policies, it can reveal useful information about the different dynamics of inequality, where inequality at the median, α, can be analyzed in parallel with that at the extremes described by β and γ.

The ABG approach relies on an easy-to-use family of distributions to model income distributions. It can be used, for example, to model extremely unequal distributions such as Zipf laws (Gabaix, 1999), which are extreme Pareto distributions with α close to 1. It also helps us to understand why the Gini coefficient can pose problems when the isograph is far from being a constant (when β and γ differ greatly from 0).

In this approach, the magnitudes of ranks and incomes, defined by logit(quantile) and log(income), are almost linearly-related. The logit(quantile) may therefore be an important tool for the measurement of inequality, and could be used in other fields such as income volatility and more generally in stratification research. The further development of the ABG should include the analysis of statistical significance and group decomposability. As the ABG coefficients come from linear regressions, we can add control variables to understand how the gaps between groups (education, gender, etc.) contribute to overall inequality. Finally, we also need further analysis of isograph shapes when the absolute value of X is over 5, for the very rich and very poor.

The results that we presented here can also be found with more traditional tools, but the α, β, γ ABG method, the CF and the isograph, and the associated strobiloid, represent more systematic and easier to use tools for the detection of particular shapes, propose better measures of the income distribution, and help us to better understand the anisotropy of inequality.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's web-site:

Appendix 1: Figure of 232 Isographs
Appendix 2: Table of 30 Inequality Indices
Appendix 2a: Comparisons of 30 indicators of inequality samples and a Principal Component Analysis PCA on 232 LIS samples
Appendix 3: General Correlation Matrix of 30 Inequality Indicators
Appendix 4: Figure of 32 Countries Strobiloids
Appendix 5: Distribution Simulator for ABG

A program for the ABG method is available to Stata users who can install the program directly by typing: ssc install gb2fit.