Inequality Analysis Tools

Conchita D'Ambrosio
conchita.dambrosio@uni.lu

Based on:

My papers:

- "Deprivation and Social Exclusion" (joint with W. Bossert and V. Peragine), *Economica*, 74, 777-803, 2007.
- "Dynamic Measures of Individual Deprivation" (joint with W. Bossert), *Social Choice and Welfare*, 28, 77-88, 2007.
- "Deprivation in the São Paulo Districts: Evidence from 2000" (joint with R. Imanishi Rodrigues), *World Development*, 36, 1094-1112, 2008.

On some notes downloaded from the web (thanks to colleagues for making them available!)

And on:

Atkinson, A.B. and A. Brandolini:
http://siteresources.worldbank.org/INTDECINEQ/Resources/1149208-1169141694589/Global_World_Inequality.pdf

Chakravarty, S.R.: "Relative Deprivation and Satisfaction Orderings", Keio Economic Studies, 34, 17-31, 1997.

Duclos, J-Y., J.M. Esteban and D. Ray, "Polarization: Concepts, Measurement, Estimation," Econometrica, 72, 1737-1772, 2004.

Esteban. J.M. and D. Ray, "On the Measurement of Polarization," Econometrica, 62, 819-851, 1994.

Hey, J.D. and P. Lambert: "Relative Deprivation and the Gini Coefficient: Comment", Quarterly Journal of Economics, 95, 567-573, 1980.

Podder, N., "Relative Deprivation, Envy and Economic Inequality," Kyklos, 3, 353-376, 1996.

Yitzhaki, S. (1979): "Relative Deprivation and the Gini Coefficient", Quarterly Journal of Economics, 93, 321-324, 1979.

Many people are talking about inequality.

Many people are studying inequality and its consequences on various outcomes, including economic growth and the crisis.

But what is inequality?

Is it really this inequality we are interested in?

In this lecture we will discuss about the above issues.

Following Andrew's presentation we can think of inequality in a:

1) normative way (Oh, there is too much inequality)

2) comparative way (Oh, there are so many people richer than me; or we are the 99%)

and measure the effects of inequality on individual behaviour.

Depending on our interests we should (or not) use the Gini coefficient.

Inequality is not only the Gini coefficient. Gini measures one particular type of inequality.

Notation

Income distribution:

$\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$) is the set of all (all non-negative, all positive) real numbers.

An income distribution is a list incomes of different individuals.

If there are n persons in the society, the incomes could be listed as $x_1, x_2, ..., x_n$ where $x_i \geq 0$ is the income of person i, with (the strict inequality) $>$ for at least one i, $1 \leq i \leq n$, and n is an arbitrary positive integer. We write $x = (x_1, x_2, ..., x_n)$. Let $\mathbb{D}$ be the space of all such distributions.

We write $\lambda(x)$ (or simply λ) for the mean of x and $m(x)$ (or simply m) for the median of x

$\overline{x}$ represents the illfare ranked permutation of x, that is $\overline{x}_1 \leq \overline{x}_2 \leq ...\overline{x}_n$.

Notation

The distinct levels of incomes are collected in a vector $(x_1, ..., x_k)$ where $k \leq n$. Let π_j indicate the population share composed of individuals experiencing the same level of income, x_j. A distribution is $(\pi, x) \equiv (\pi_1, ..., \pi_k; x_1, ..., x_k)$, $x_i \neq x_j$ for all $i, j \in \{1, ..., k\}$. Let Ω be the space of all distributions. $\overline{x}$ indicates the illfare ranked permutation of the vector x.

Notation

Functioning failures distribution:

The distinct levels of functioning failures are collected in a vector $(q_1, ..., q_k)$ where $k \leq n$.. Let π_j indicate the population share composed of individuals with the same level of functioning failures, q_j. A distribution is $(\pi, q) \equiv (\pi_1, ..., \pi_k; q_1, ..., q_k)$, $q_i \neq q_j$ for all $i, j \in \{1, ..., k\}$. Let Θ be the space of all distributions. $\overline{q}$ indicates the illfare ranked permutation of the vector q.

Inequality Measures

Definition

An inequality measure is a function I from D to R which, for each distribution x in D indicates the level I(x) of inequality in the distribution.

Four Basic Properties

Definition

We say that x is obtained from y by a *permutation of incomes* if x = Py, where P is a permutation matrix.

Ex

$$x = Py = \begin{bmatrix} 010 \\ 100 \\ 001 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$$

Symmetry (Anonymity)

If x is obtained from y by a permutation of incomes, then I(x)=I(y).

All differences across people have been accounted for in x

Def

We say that x is obtained from y by a *replication* if the incomes in x are simply the incomes in y repeated a finite number of times

Ex

$$x = (y_1, y_1, y_2, y_2, \ldots\ldots, y_n, y_n)$$

$$x = (6, 6, 6, 1, 1, 1, 8, 8, 8)$$

Replication Invariance (Population Principle)

If x is obtained from y by a replication, then I(x)=I(y).

Can compare across different sized populations

Def

We say that x is obtained from y by a *proportional change* if x=αy, for some α > 0.

Ex

$$y = (6, 1, 8) \qquad x = (12, 2, 16)$$

Scale Invariance (Zero-Degree Homogeneity)

If x is obtained from y by a proportional change, then I(x)=I(y).

Relative inequality

Def

We say that x is obtained from y by a *(Pigou-Dalton) regressive transfer* if for some i, j:

i) $y_i \leq y_j$

ii) $y_i - x_i = x_j - y_j > 0$

iii) $x_k = y_k$ for all k different to i,j

Ex

$$y = (2, 6, 7) \qquad x = (1, 6, 8)$$

Transfer Principle

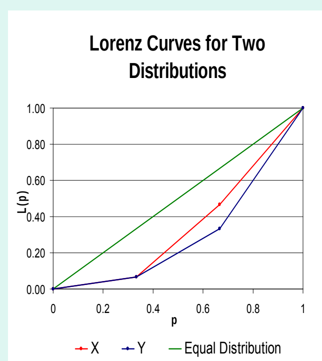
If x is obtained from y by a regressive transfer, then I(x) > I(y).

The Lorenz Curve and the Four Axioms

Symmetry and *Replication invariance* satisfied since permutations and replications leave the curve unchanged.

Proportional changes in incomes do not affect the LC, since it is normalized by the mean income. Only *shares* matter. So it is *scale invariant*.

A regressive transfer will move the Lorenz curve further away from the diagonal. So it satisfies *transfer principle*.



$$y = (1, 5, 9)$$
$$x = (1, 6, 8)$$

Lorenz Consistency

Def

An inequality measure *I: D→R* is *Lorenz consistent* whenever the following hold for any *x* and *y* in D:

(i) if x Lorenz dominates y, then I(x) < I(y), and

(ii) if x has the same Lorenz curve as y, then I(x) = I(y).

Theorem

An inequality measure *I(x)* is Lorenz consistent if and only if it satisfies symmetry, replication invariance, scale invariance and the transfer principle.

Note

If Lorenz curves don't cross, then all relative measures follow the Lorenz curve.

If Lorenz curves cross, then some relative measure of inequality might be used to make the comparison. But the judgment may depend on the chosen measure.

The Gini coefficient is Lorenz consistent.

When you use the Gini coefficient this is the type of inequality you are thinking of.

Thinking about inequality

Amiel and Cowell, 1999, CUP

In each of the first nine questions you are asked to compare two distributions of income. Please state which of them you consider to be the **more unequally** distributed by circling A or B. If you consider that both of the distributions have the same inequality then circle both A and B.

1) A = (5, 8, 10) B = (10, 16, 20)

2) A = (5, 8, 10) B = (10, 13, 15)

3) A = (5, 8, 10) B = (5, 5, 8, 8, 10, 10)

4) A = (1, 4, 7, 10, 13) B = (1, 5, 6, 10, 13)

Inequality and proportionate and absolute income differences (% responses) (N=1108)

Numerical problems

		Add 5 units		
		Down	Up	Same
Double income	Down	8	2	5
	Up	15	3	17
	Same	37	5	9

Verbal questions

		Add 5 units		
		Down	Up	Same
Double income	Down	7	1	4
	Up	21	2	17
	Same	30	3	14

The effect on inequality of cloning the distributions (% responses) (N=1108)

	Numerical	Verbal
Down	31	22
Up	10	9
Same	58	66

The transfer principle (% responses) (N=1108)

	Numerical	Verbal
Agree	35	60
Strongly disagree	42	24
Disagree	22	14

Agree=A is more unequal than B
Strongly Disagree=B is more unequal than A
Disagree=A and B have the same inequality

What happens when we depart from scale invariance?

GLOBAL WORLD INEQUALITY: ABSOLUTE, RELATIVE OR INTERMEDIATE?

Anthony B. Atkinson
and
Andrea Brandolini

Aim

This paper examines how the conclusions on the evolution of world income inequality might be affected by abandoning the relative inequality criterion.

In particular:

- examine **methodological issues** and discuss classes of measures that combine the relative and absolute criterion.
- present the **results** from applying these different measures to the distribution of income in the **world**.
  - first discuss **international inequality**;
  - then give illustrative results on **global inequality.**

In particular:

- examine **methodological issues** and discuss classes of measures that combine the relative and absolute criterion.

- present the **results** from applying these different measures to the distribution of income in the **world**.
  - first discuss **international inequality**;
  - then give illustrative results on **global inequality.**

"global" differs from "international" in that within-country inequality is accounted for.

Question:

How shall we distribute/take a given sum of money within/from the population so that income inequality remains unchanged?

The answer social scientists generally give is:

"income inequality remains unchanged when all incomes are increased/decreased by the same **proportion**".

They believe in **scale invariance**.

Inequality indices, *I*, are **relative**.

The answer social scientists generally give is:

"income inequality remains unchanged when all incomes are increased/decreased by the same **proportion**".

I(10, 20, 30) = I(5, 10, 15) = I(20, 40, 60)

I(x) = I(cx) for all $c>0$, homogeneity of degree zero.

They believe in **scale invariance**.

Inequality indices, *I*, are **relative**.

Are social scientists correct?

It depends.

Other answers can be given to the same question.

Alternatives:

“Income inequality remains unchanged when all incomes are increased/decreased by the same **absolute** amount”.

They believe in **translation invariance**.
Inequality indices, I, used are **absolute**.

Alternatives:

"Income inequality remains unchanged when all incomes are increased/decreased by the same **absolute** amount".

I(10, 20, 30) = *I*(0, 10, 20) = *I*(15, 25, 35)

$I(x) = I(x+t1^n)$ for all $t>0$.

They believe in **translation invariance**.

Inequality indices, *I*, used are **absolute**.

Alternatives:

"Income inequality remains unchanged when some kind of **combination** between an equal-proportion and an equal absolute amount increase/decrease of all incomes is performed".

They take a middle stand and believe that an equal-proportion distribution increases inequality, while an equal-absolute amount distribution decreases inequality (“compromise property”).

Inequality indices, *I*, used are **intermediate**.

The invariance condition of Bossert and Pfingsten (1990) is:

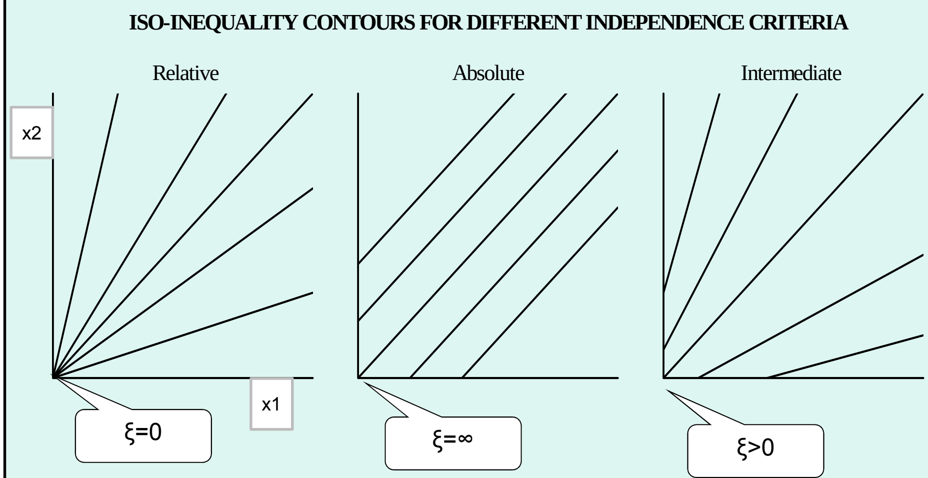
$I(x) = I(a[x+\xi 1^n]-\xi 1^n)$ for all $a>1$, where $\xi>0$ is a parameter indicating the inequality concept, value judgment parameter.

similar to Kolm’s (1976) invariance condition

$sI(x) = I(s[x+m1^n]-m1^n])$ for all $s>0$, where $m>0$ is a parameter indicating the inequality concept, value judgment parameter.

What is ξ of Bossert and Pfingsten?

ξ is a parameter indicating the inequality concept, value judgment parameter, absolute value of origin of rays.



There is no single correct answer to the distribution/taxation question posted above, the aforementioned views reflect value judgment in measuring income inequality.

In order to obtain reasonable inequality rankings, it may be desirable for different views of value judgment to be consulted in assessing income inequality.

Caveat: the inequality value of a population remains unchanged when incomes are measured in different currency units only for relative measures.

Results

Relative indices: the mean logarithmic deviation, the Gini index and the Theil index.

Absolute indices: absolute Gini index and the Kolm index for different values of its parameter.

Intermediate indices: Kolm, and Bossert and Pfingsten for different values of its parameters.

International income inequality

It examines the “international” rather than the “global” distribution of income since they study differences across countries in **per capita GDP** weighing each observation by the country’s **population**, but making no allowance for the distribution of income within the country.

Use real per capita GDP and population size for all countries and years in the period 1970-2000 for which both variables are available from the Penn World Table, Version 6.1 (Heston, Summers and Aten, 2002).

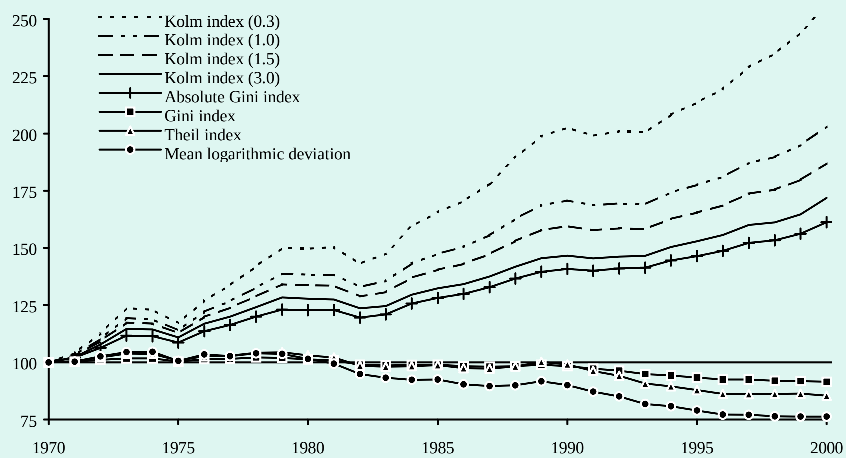
Use *real* incomes expressed in U.S. constant dollars.

Full sample comprises 152 countries, but not all countries have a continuous run of data from 1970 to 2000: there are 30 or 31 observations for 106 countries, between 21 and 29 for another 27, and 15 or less for the remaining 29.

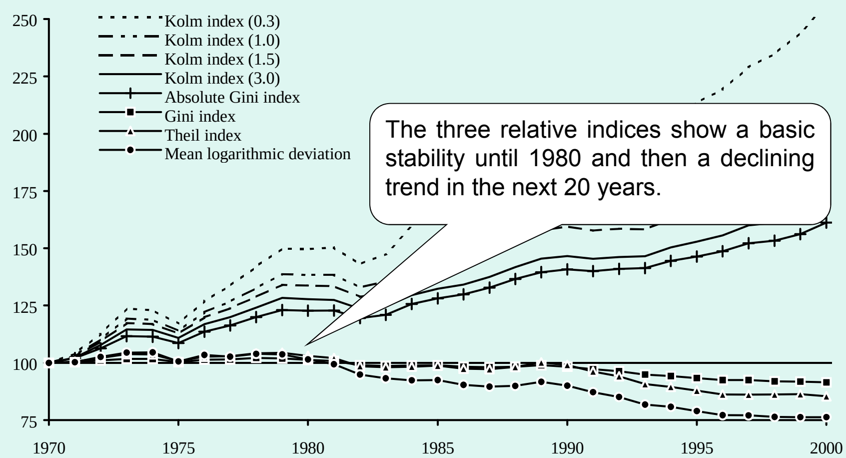
To avoid that measured trends reflect changes in country coverage, they concentrate on the sub-sample composed of the **106 countries** with 30 or 31 observations.

It includes **27** of the 30 countries which are currently member of the **OECD** (the Czech Republic, Poland and the Slovak Republic being those excluded), and all the **most populous nations but** for Russia and Vietnam (i.e. China, India, Indonesia, Brazil, Pakistan, Nigeria, Philippines, Thailand, Iran, Egypt, Ethiopia).

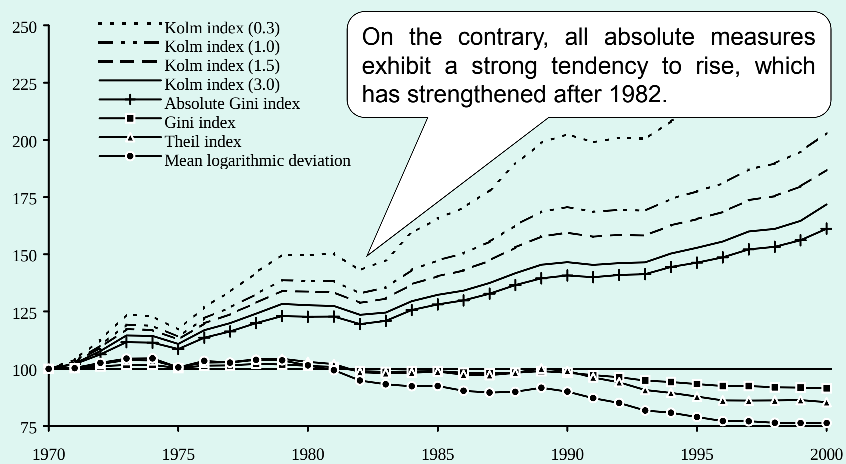
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
RELATIVE AND ABSOLUTE INDICES
(Indices: 1970=100)

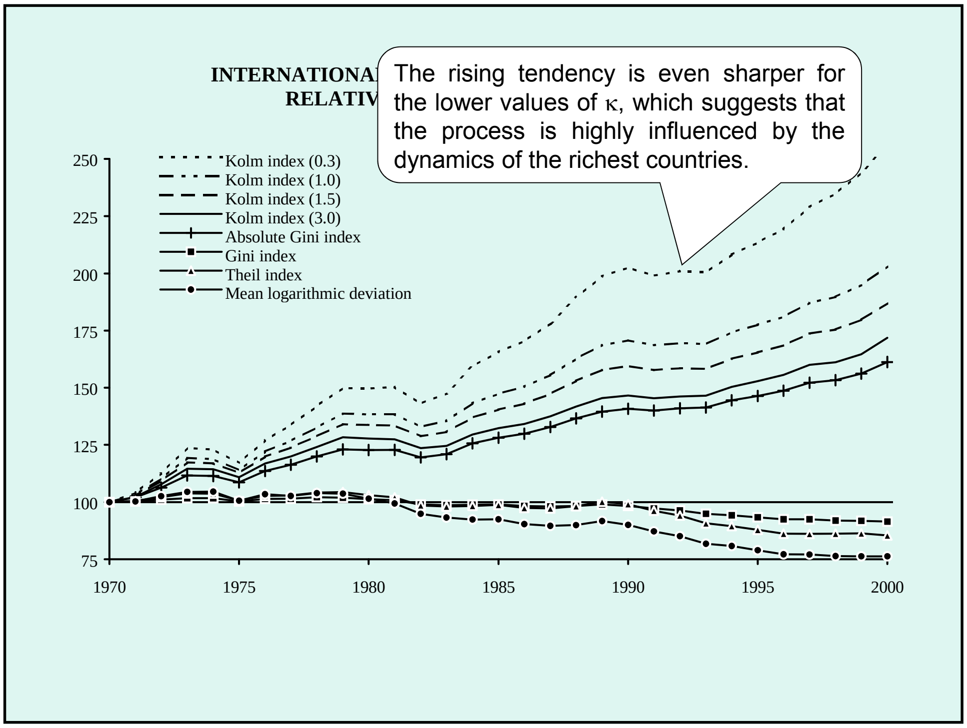


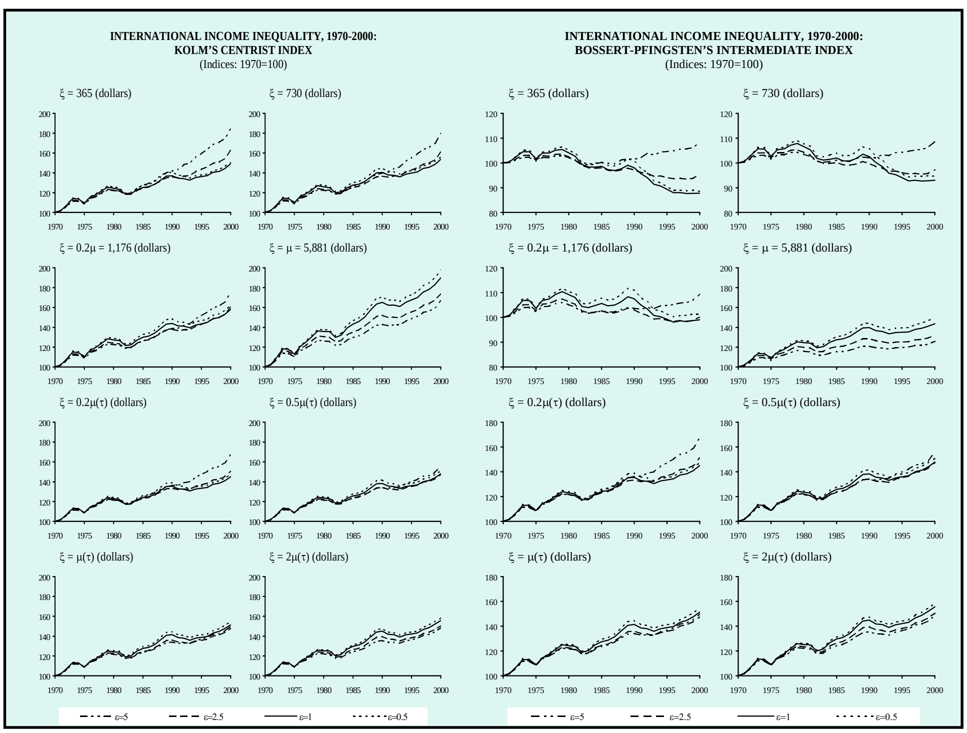
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
RELATIVE AND ABSOLUTE INDICES
(Indices: 1970=100)

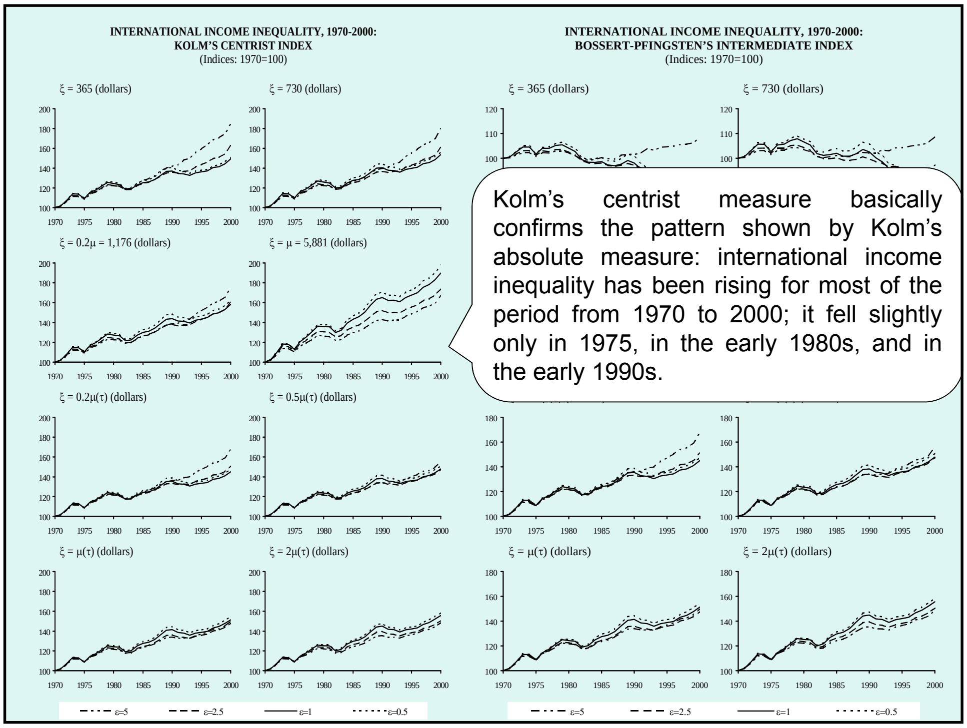


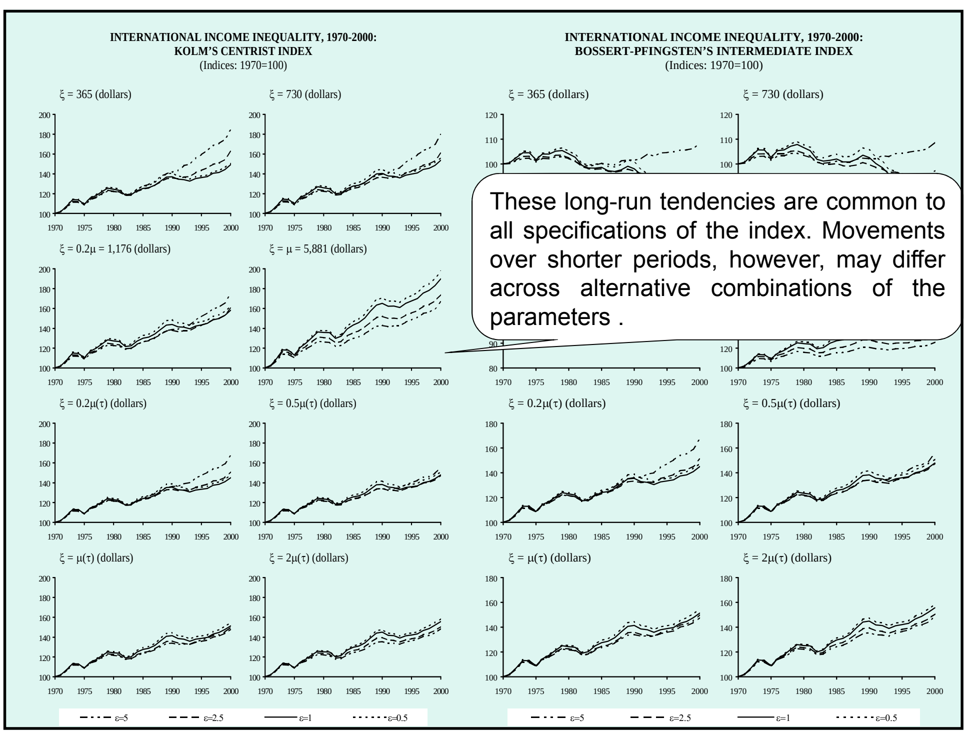
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
RELATIVE AND ABSOLUTE INDICES
(Indices: 1970=100)




INTERNATIONAL
RELATIV
The rising tendency is even sharper for the lower values of κ, which suggests that the process is highly influenced by the dynamics of the richest countries.
Kolm index (0.3)
Kolm index (1.0)
Kolm index (1.5)
Kolm index (3.0)
Absolute Gini index
Gini index
Theil index
Mean logarithmic deviation
250
225
200
175
150
125
100
75
1970
1975
1980
1985
1990
1995
2000


INTERNATIONAL INCOME INEQUALITY, 1970-2000:
KOLM'S CENTRIST INDEX
(Indices: 1970=100)
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
BOSSERT-PFINGSTEN'S INTERMEDIATE INDEX
(Indices: 1970=100)
ξ = 365 (dollars)
ξ = 730 (dollars)
ξ = 0.2μ = 1,176 (dollars)
ξ = μ = 5,881 (dollars)
ξ = 0.2μ(τ) (dollars)
ξ = 0.5μ(τ) (dollars)
ξ = μ(τ) (dollars)
ξ = 2μ(τ) (dollars)
ε=5
ε=2.5
ε=1
ε=0.5


INTERNATIONAL INCOME INEQUALITY, 1970-2000:
KOLM'S CENTRIST INDEX
(Indices: 1970=100)
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
BOSSERT-PFINGSTEN'S INTERMEDIATE INDEX
(Indices: 1970=100)
ξ = 365 (dollars)
ξ = 730 (dollars)
ξ = 0.2μ = 1,176 (dollars)
ξ = μ = 5,881 (dollars)
ξ = 0.2μ(τ) (dollars)
ξ = 0.5μ(τ) (dollars)
ξ = μ(τ) (dollars)
ξ = 2μ(τ) (dollars)
ε=5 ε=2.5 ε=1 ε=0.5
Kolm's centrist measure basically confirms the pattern shown by Kolm's absolute measure: international income inequality has been rising for most of the period from 1970 to 2000; it fell slightly only in 1975, in the early 1980s, and in the early 1990s.


INTERNATIONAL INCOME INEQUALITY, 1970-2000:
KOLM'S CENTRIST INDEX
(Indices: 1970=100)
INTERNATIONAL INCOME INEQUALITY, 1970-2000:
BOSSERT-PFINGSTEN'S INTERMEDIATE INDEX
(Indices: 1970=100)
ξ = 365 (dollars)
ξ = 730 (dollars)
ξ = 0.2μ = 1,176 (dollars)
ξ = μ = 5,881 (dollars)
ξ = 0.2μ(τ) (dollars)
ξ = 0.5μ(τ) (dollars)
ξ = μ(τ) (dollars)
ξ = 2μ(τ) (dollars)
ε=5 ε=2.5 ε=1 ε=0.5
These long-run tendencies are common to all specifications of the index. Movements over shorter periods, however, may differ across alternative combinations of the parameters .

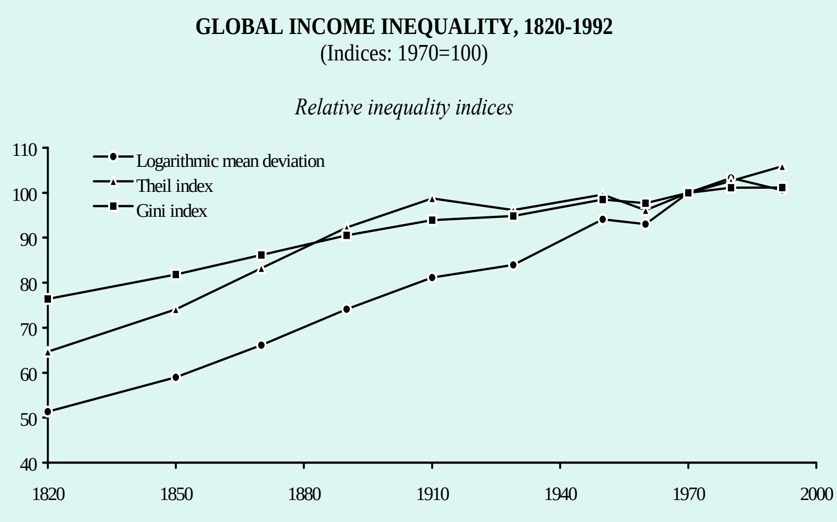
Global income inequality

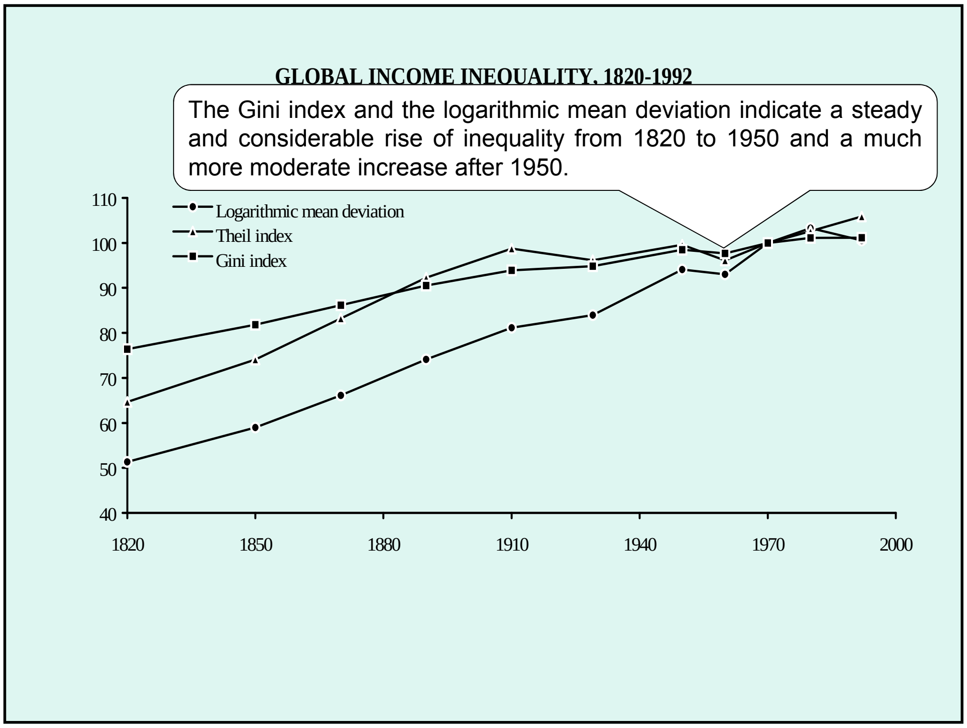
A-B try to bring in **within-country inequality**.

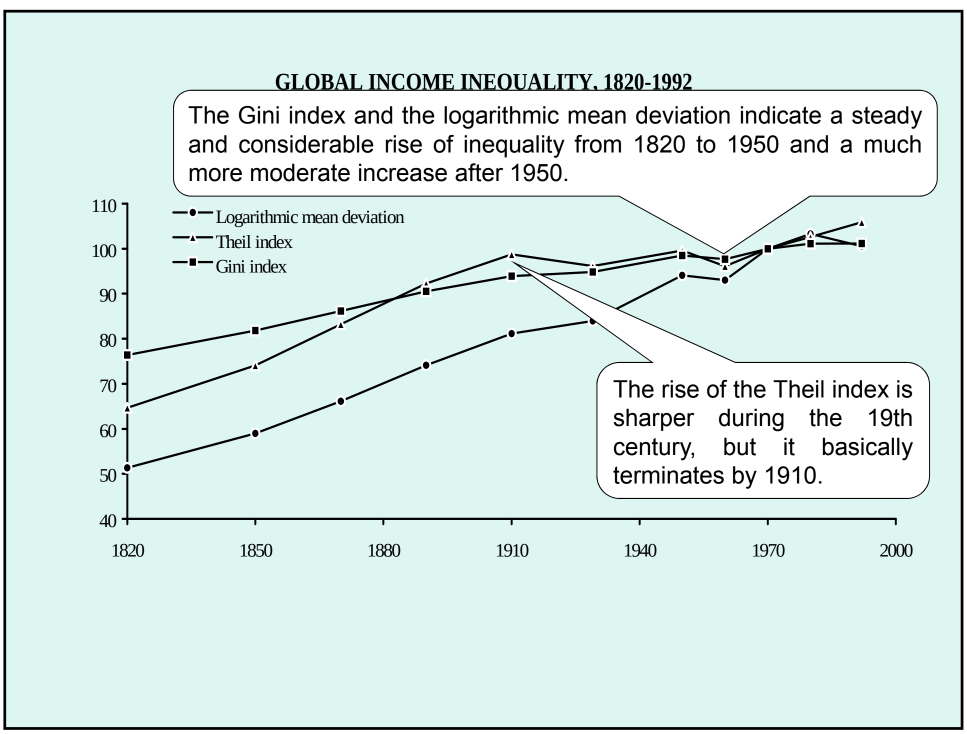
The data for the world distribution of income are those constructed by **Bourguignon and Morrisson** (2002).

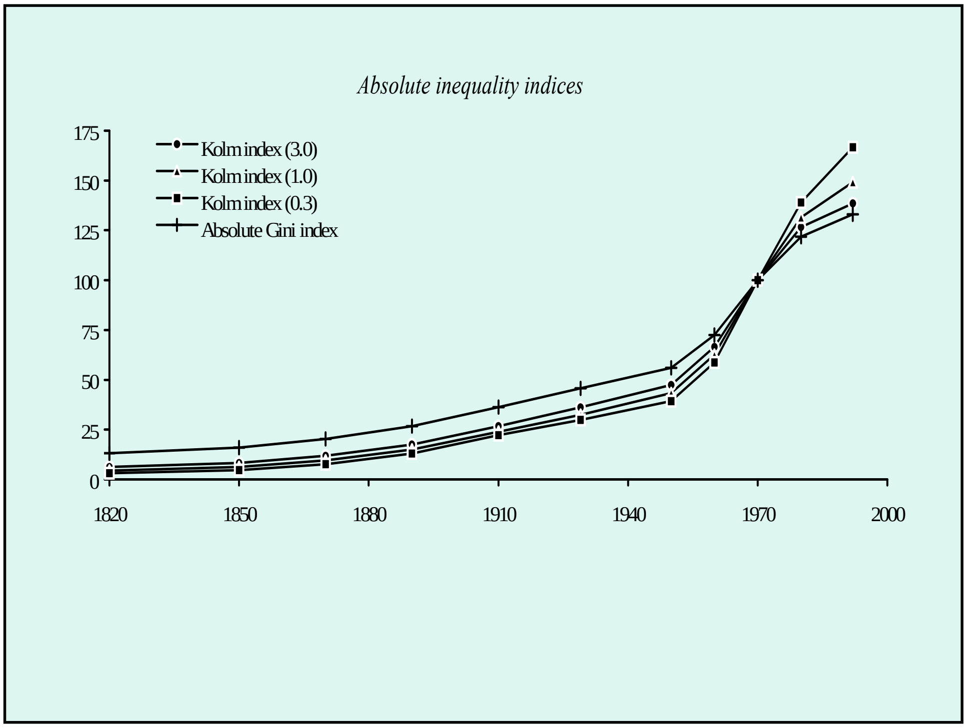
Their method is to use evidence on the national distribution (or the distribution for a grouping of countries) about the **income shares of decile groups, and the top 5 per cent**. The groups are treated as homogeneous, which means that the degree of overall inequality is under-stated, but their data provide a valuable starting point.

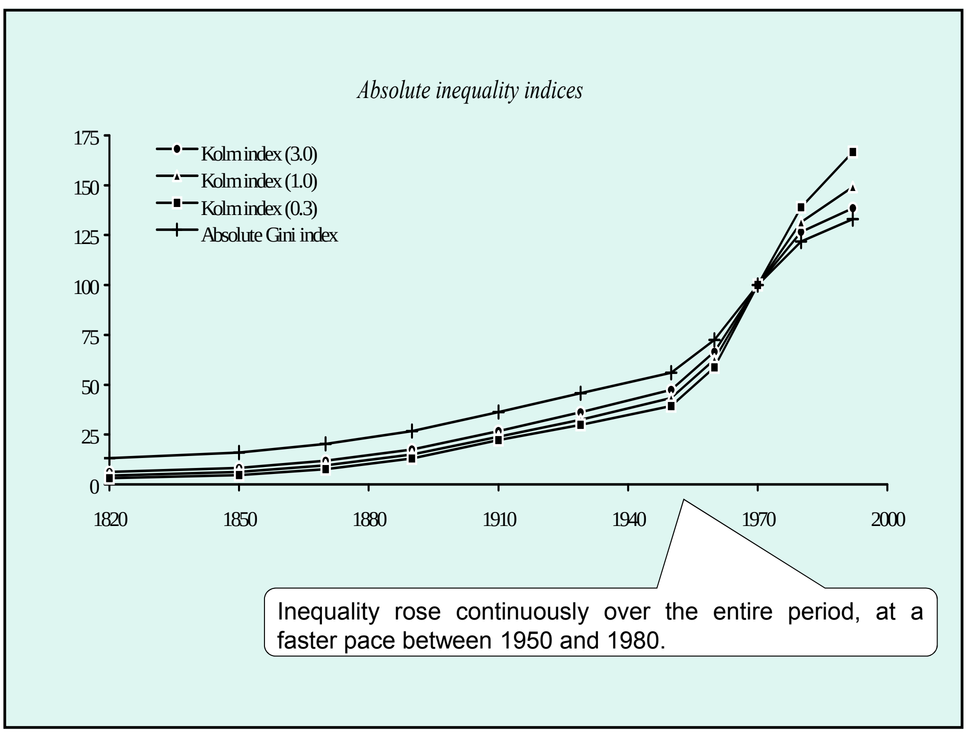
The distributional data are then combined with estimates of **national GDP per head**, expressed in constant purchasing power parity dollars (at 1990 prices), which are in turn derived from the historical time series constructed by **Maddison** (1995).

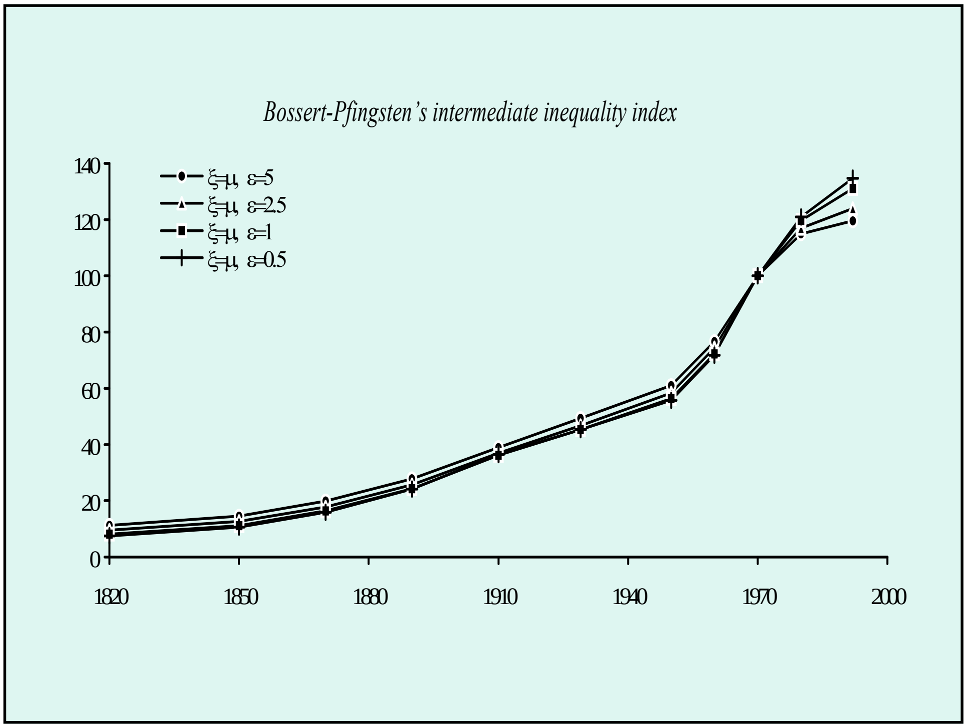


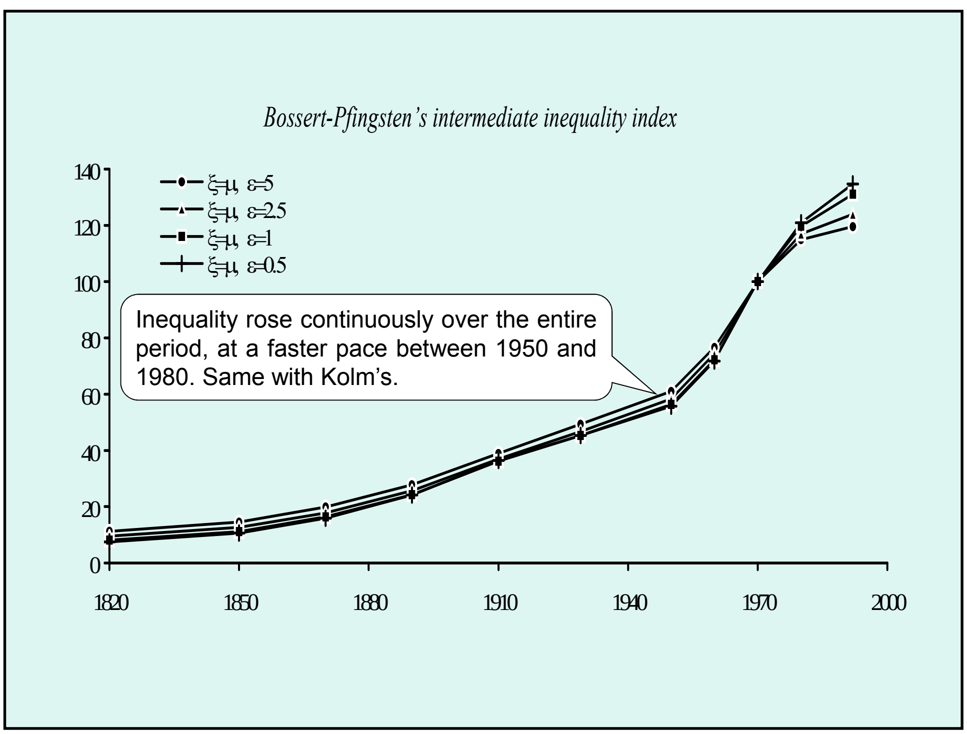

GLOBAL INCOME INEQUALITY, 1820-1992
The Gini index and the logarithmic mean deviation indicate a steady and considerable rise of inequality from 1820 to 1950 and a much more moderate increase after 1950.
Logarithmic mean deviation
Theil index
Gini index
110
100
90
80
70
60
50
40
1820
1850
1880
1910
1940
1970
2000


GLOBAL INCOME INEQUALITY, 1820-1992
The Gini index and the logarithmic mean deviation indicate a steady and considerable rise of inequality from 1820 to 1950 and a much more moderate increase after 1950.
Logarithmic mean deviation
Theil index
Gini index
The rise of the Theil index is sharper during the 19th century, but it basically terminates by 1910.
110
100
90
80
70
60
50
40
1820
1850
1880
1910
1940
1970
2000


Absolute inequality indices
Kolm index (3.0)
Kolm index (1.0)
Kolm index (0.3)
Absolute Gini index
175
150
125
100
75
50
25
0
1820
1850
1880
1910
1940
1970
2000


Absolute inequality indices
Kolm index (3.0)
Kolm index (1.0)
Kolm index (0.3)
Absolute Gini index
175
150
125
100
75
50
25
0
1820
1850
1880
1910
1940
1970
2000
Inequality rose continuously over the entire period, at a faster pace between 1950 and 1980.


Bossert-Pfingsten's intermediate inequality index
ξ=μ, ε=5
ξ=μ, ε=2.5
ξ=μ, ε=1
ξ=μ, ε=0.5
140
120
100
80
60
40
20
0
1820
1850
1880
1910
1940
1970
2000


Bossert-Pfingsten's intermediate inequality index
ξ=μ, ε=5
ξ=μ, ε=2.5
ξ=μ, ε=1
ξ=μ, ε=0.5
Inequality rose continuously over the entire period, at a faster pace between 1950 and 1980. Same with Kolm's.
140
120
100
80
60
40
20
0
1820
1850
1880
1910
1940
1970
2000

The secular movement of the world income distribution does not change whether we look at relative or non-relative measures – inequality has been rising.

The story is somewhat different, however, after the Second World War: the modest positive slope of relative inequality is matched by a steep ascent of absolute and intermediate inequality.

Conclusion: international inequality

The international distribution of real per capita GDP (i.e. ignoring within-country disparities) narrowed from 1970 to 2000 if we adopt a relative view of inequality;

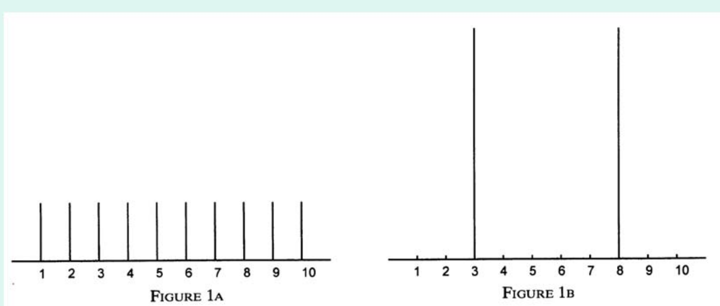
it widened considerably if we assume an absolute or an intermediate conception, regardless of the index chosen and for most of the values of parameters.

Only the Bossert and Pfingsten's index for some combinations of the parameters suggests a fall of intermediate inequality.

What happens when we depart from Pigou-Dalton?

Probably what influences behaviour of people is not relative inequality. There are many more indices that capture differences in income. Let us see a few.

You are interested in measuring the evolution of the middle class



Do not use a measure of inequality!

Polarization

Polarization is different from inequality:

It fails to satisfy Pigou-Dalton transfers principle.

Polarization

Polarization is different from inequality:

It fails to satisfy Pigou-Dalton transfers principle.

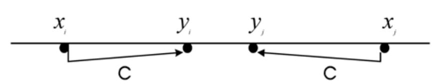


Figure 1: Progressive transfer.

Axiom 1 : *Pigou-Dalton Transfers Principle (PD). If $y \in \mathbb{D}^n$ is obtained from $x \in \mathbb{D}^n$ by a progressive transfer, then $I^n(y) < I^n(x)$.*

Thus, *PD* implies that a progressive transfer reduces inequality. Likewise, a regressive transfer increases inequality.

Polarization

Polarization is different from inequality:

It fails to satisfy Pigou-Dalton transfers principle.

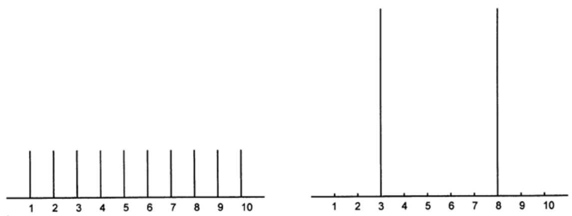


FIGURE 1A FIGURE 1B

Polarization

Polarization is different from inequality:

It fails to satisfy Pigou-Dalton transfers principle.

Inequality decreased
Polarization increased

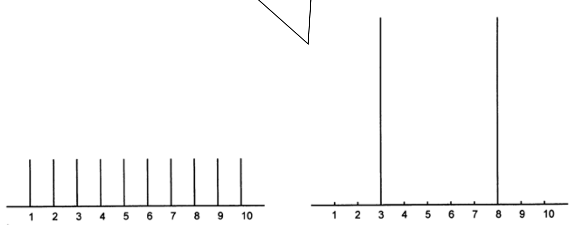


FIGURE 1A FIGURE 1B

To measure polarization you can follow two approaches:

1) **Esteban and Ray;**
2) **Wolfson.**

They are different variations of the Gini coefficient.

Inequality in Gini

The most well-known index of inequality is the Gini coefficient defined as:

$$G(x) = \frac{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|}{2\lambda(x)}. \tag{1}$$

The numerator of (1) is the Gini mean difference. When divided by the mean $\lambda(x)$ it becomes the relative mean difference. Since

$$\min(x_i, x_j) = \frac{x_i + x_j - |x_i - x_j|}{2}, \tag{2}$$

we can rewrite $G^n(x)$ as

$$\begin{aligned} G^n(x) &= 1 - \tfrac{1}{n^2\lambda(x)}\sum_{j=1}^{n}\sum_{i=1}^{n}\min(x_i, x_j) \\ &= 1 - \tfrac{1}{n^2\lambda(x)}\sum_{i=1}^{n}(2(n-i)+1)\overline{x}_i. \end{aligned} \tag{3}$$

Inequality in Gini

Each individual feels alienated from others located at different points of the income scale:

$$\mathring{A}_i(x) = \sum_{j=1}^{n} |x_j - x_i|$$

if there is more than one individual with the same income level:

$$A_i(\pi, x) = \sum_{j=1}^{k} |x_j - x_i| \pi_j$$

Inequality in Gini

Income inequality, in the whole society, is the sum of these sentiments of alienation:

$$I(\pi, x) = \sum_{i=1}^{k} \pi_i A_i(\pi, x) = \sum_{i=1}^{k} \sum_{j=1}^{k} \pi_j \pi_i |x_j - x_i|$$

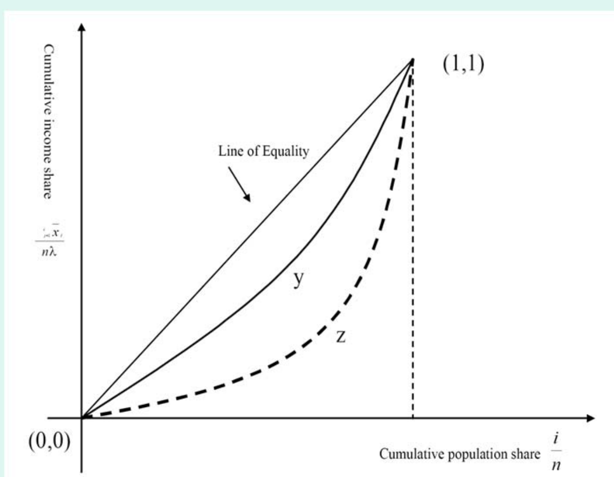
Inequality in Gini

Income inequality, in the whole society, is the sum of these sentiments of alienation:

$$I(\pi, x) = \sum_{i=1}^{k} \pi_i A_i(\pi, x) = \sum_{i=1}^{k} \sum_{j=1}^{k} \pi_j \pi_i |x_j - x_i|$$

Proportional to Absolute Gini

Lorenz Curve



The Lorenz Curve

Polarization: the ER Approach

Each individual feels alienated from others located at different points of the income scale:

$$\mathring{A}_i(x) = \sum_{j=1}^{n} |x_j - x_i|$$

if there is more than one individual with the same income level:

$$A_i(\pi, x) = \sum_{j=1}^{k} |x_j - x_i| \pi_j$$

Polarization: the ER Approach

Each individual identifies with people having the same income, identification/alienation gives rise to effective alienation:

$$EA_i(\pi, x) = \pi_i^{\alpha} A_i(\pi, x)$$

Polarization, in the whole society, is the sum of these sentiments of effective alienation:

$$P(\pi, x) = \sum_{i=1}^{k} \pi_i EA_i(\pi, x) = \sum_{i=1}^{k} \sum_{j=1}^{k} \pi_i^{1+\alpha} \pi_j |x_j - x_i|$$

Polarization: the ER Approach

Each individual identifies with people having the same income, identification/alienation gives rise to effective alienation:

$$EA$$

The Esteban-Ray (absolute) measure
The Duclos-Esteban-Ray measure

Polarization in the whole society, is the sum of these sentiments of effective alienation:

$$P(\pi, x) = \sum_{i=1}^{k} \pi_i EA_i(\pi, x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \pi_i^{1+\alpha}\pi_j |x_j - x_i|$$

Polarization: the Wolfson's approach

Two characteristics that are regarded as being intrinsic to the notion of polarization:

1. increasing spread,
2. increasing bipolarity.

Polarization: the Wolfson's approach

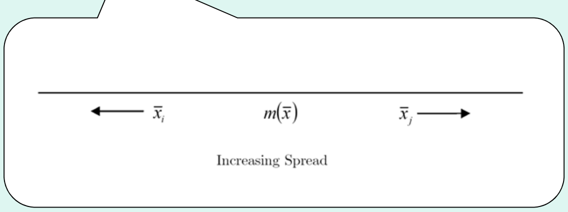
According to increasing spread, a movement of incomes from the middle position to the tails of the income distribution increases polarization.

In other words, as the distribution becomes more spread out from the middle position, polarization increases.

Polarization: the Wolfson's approach

According to increasing spread, a movement of incomes from the middle position to the tails of the income distribution increases polarization.

In other words, as the distribution becomes more spread out from the middle position, polarization increases.

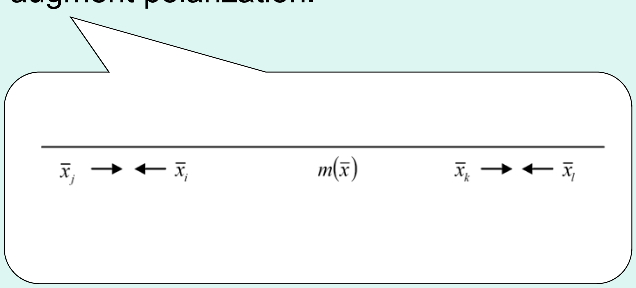


Polarization: the Wolfson's approach

On the other hand, increasing bipolarity means that a clustering of incomes below or above the median augment polarization.

Polarization: the Wolfson's approach

On the other hand, increasing bipolarity means that a clustering of incomes below or above the median augment polarization.



Polarization: the Wolfson's approach

The measure of polarization in Wolfson (1994) can be rewritten as

$$P^W = \frac{\lambda}{m}\left[\frac{1}{2} - L - \frac{G}{2}\right],$$

where m stands for the median income, λ for the mean income, G for the Gini index and $L = L\left(\frac{1}{2}\right)$ for the value of the ordinate of the Lorenz curve at the median income.

Polarization: the Wolfson's approach

The measure of polarization in Wolfson (1994) can be rewritten as

$$P^W = \frac{\lambda}{m}\left[\frac{1}{2} - L - \frac{G}{2}\right],$$

where m stands for the median income, λ for the mean income, G for the Gini index and $L = L\left(\frac{1}{2}\right)$ for the value of the ordinate of the Lorenz curve at the median income.

Class of indices by Wang and Tsui (JPET, 2000)

Polarization: the Wolfson's approach

Wang and Tsui (JPET, 2000) suggested the use of the following as absolute and relative indices of polarization respectively:

$$P_{\Phi}(x) = \frac{1}{n}\sum_{i=1}^{n}\Phi(d_i),$$

$$P_{\Psi}(x) = \frac{1}{n}\sum_{i=1}^{n}\Psi(D_i).$$

where:

$$d_i = |x_i - m(x)|,$$

and

$$D_i = \left|\frac{x_i - m(x)}{m(x)}\right|.$$

d_i is translation invariant while D_i is scale invariant. Φ and Ψ are increasing, strictly concave in $\mathbb{R}_+$ and $\Phi(0) = 0$ and $\Psi(0) = 0$.

Polarization curve

The (relative) polarization curve of any income distribution shows for any population proportion, how far the total income enjoyed by that proportion, expressed as a fraction of $nm(x)$, is from the corresponding income that would receive under the hypothetical distribution where everybody enjoys the median income.

For any $x \in \mathbb{D}$, the polarization curve (PC) ordinate corresponding to the population proportion $\frac{k}{n}$ $(1 \leq k \leq \overline{n})$ is $P\left(x, \frac{k}{n}\right) = \frac{1}{nm(x)} \sum_{k \leq i \leq \overline{n}}^{n} (m(x) - x_i)$, and corresponding to the population proportion $\frac{k}{n}$, $(\overline{n} \leq k \leq n)$ this ordinate is $P\left(x, \frac{k}{n}\right) = \frac{1}{nm(x)} \sum_{\overline{n} \leq i \leq k}^{n} (x_i - m(x))$, where $\overline{n} = \frac{n+1}{2}$.

Note that the ordinate at $\frac{\overline{n}}{n}$ involves the income level $x_{\overline{n}} = m(x)$. Now, if n is odd, $m(x)$ is one of the incomes in the distribution. However, for even n, $x_{\overline{n}}$ is not in x, we define the ordinate at $\frac{\overline{n}}{n}$, since in polarization measurement, the median income is the reference income.

Polarization curve

Example 3 : *For the distributions* $x = (1, 3, 5, 9, 11)$, $m(x) = 5$, $x_- = (1, 3)$, $x_+ = (9, 11)$. *The ordinates of the polarization curve are:*

$$P\left(x, \frac{1}{5}\right) = \tfrac{1}{25}\left((5-1)+(5-3)\right) = \tfrac{6}{25};$$

$$P\left(x, \frac{2}{5}\right) = \tfrac{1}{25}\left((5-3)\right) = \tfrac{2}{25};$$

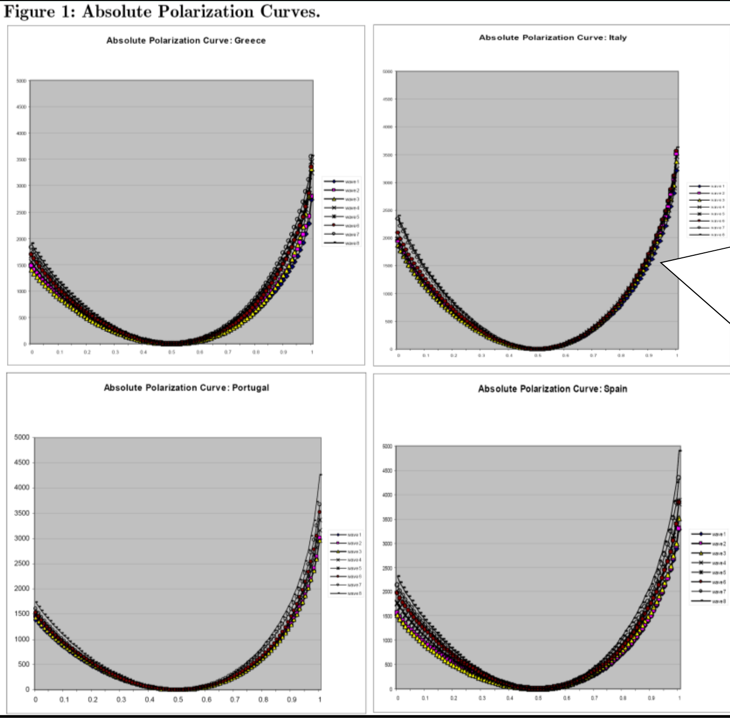
$$P\left(x, \frac{3}{5}\right) = 0;$$

$$P\left(x, \frac{4}{5}\right) = \tfrac{1}{25}\left((9-5)\right) = \tfrac{4}{25};$$

$$P\left(x, \frac{5}{5}\right) = \tfrac{1}{25}\left((9-5)+(11-5)\right) = \tfrac{10}{25}.$$

For a typical income distribution x, up to $\frac{\overline{n}}{n}$, the polarization curve decreases monotonically, at $\frac{\overline{n}}{n}$ it coincides with the horizontal axis and then it increases monotonically. If x is an equal distribution, then the curve becomes the horizontal axis itself.

Figure 1: Absolute Polarization Curves.



An asymmetry in distances from the median exists in all cases.

This observation is a consequence of the longer right tail of the curves.

Polarization: the Wolfson's approach

Definition 2 : *Given any two income distributions $x, y \in \mathbb{D}$, x is said to dominate y with respect to polarization, which we write xPy if the polarization curve of x is nowhere below that of y, and at some places above.*

Theorem 11 : *Let $x,y \in \mathbb{D}$ be arbitrary. Then the following statements are equivalent:*
1) xPy;
2) $P(x) > P(y)$ for all relative polarization indices belonging to the class of Wang and Tsui.

This theorem indicates that an unambiguous ranking of income distribution can be obtained if and only if their polarization curves do not intersect.

You are interested in understanding the effects of feeling poorer than others.

Use a measure of deprivation!

(For a society it can be the Gini coefficient but for an individual is not. And there are also other measures of deprivation)

Deprivation

The definition of relative deprivation adopted is the following:

"We can roughly say that [a person] is relatively deprived of X when

(i) he does not have X,

(ii) he sees some other person or persons, which may include himself at some previous or expected time, as having X,

(iii) he wants X, and

(iv) he sees it as feasible that he should have X"

(Runciman, 1966, p.10).

Runciman further adds: "The magnitude of relative deprivation is the extent of the difference between the desired situation and that of the person desiring it".

Runciman further adds: “The magnitude of relative deprivation is the extent of the difference between the desired situation and that of the person desiring it”.

One of the key variables in measuring deprivation is the **reference group**, that is the group with which a person compares itself.

Deprivation

Each individual feels deprived only in comparison with others located at higher points of the income scale:

$$d_i(x) = (x_j - x_i) \quad \text{if } x_i < x_j$$
$$= 0 \quad \text{else}$$

Deprivation

Each individual feels deprived only in comparison with others located at higher points of the income scale:

$$d_i(x) = (x_j - x_i) \quad \text{if } x_i < x_j$$
$$= 0 \quad \text{else}$$

Comparison with others located at lower points of the income scale gives rise to "Satisfaction"

Deprivation

Total deprivation felt by an individual is:

$$D_i(x) = \sum_{j=i+1}^{k} (\overline{x}_j - \overline{x}_i)\,\pi_j$$

$$D_i(x) = \frac{\sum_{j=i+1}^{n} (\overline{x}_j - \overline{x}_i)}{n}$$

Deprivation, in the whole society, is the sum of these sentiments:

$$D(x) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} (\overline{x}_j - \overline{x}_i)\,\pi_j \pi_i$$

$$D(x) = \frac{\sum_{i=1}^{n} \sum_{j=i+1}^{n} (\overline{x}_j - \overline{x}_i)}{n^2}$$

Deprivation

Total deprivation felt by an individual is:

$$D_i(x) = \sum_{j=i+1}^{k} (\overline{x}_j - \overline{x}_i)\,\pi_j$$

$$D_i(x) = \frac{\sum_{j=i+1}^{n} (\overline{x}_j - \overline{x}_i)}{n}$$

The Yitzhaki measure
which is equal to the Absolute Gini

Deprivation … he sum of these sentiments:

$$D(x) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} (\overline{x}_j - \overline{x}_i)\,\pi_j \pi_i$$

$$D(x) = \frac{\sum_{i=1}^{n} \sum_{j=i+1}^{n} (\overline{x}_j - \overline{x}_i)}{n^2}$$

Deprivation curve

Kakwani (1984) introduced the relative deprivation curve. The area under the deprivation curve is the **Gini coefficient**, the index of relative deprivation.

Deprivation curve

Following Chakravarty, the total relative deprivation felt by an individual is:

$$D_i^r(x) = \frac{\sum_{j=i+1}^{n} (\overline{x}_j - \overline{x}_i)}{n\lambda(x)} .$$

We can rewrite $D_i^r(x)$

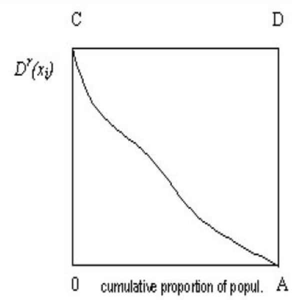
Ordinate of Lorenz Curve

$$D_i^r(x) = 1 - L(x_i) - \frac{(n-i)\,\overline{x}_i}{n\lambda(x)},$$

Deprivation curve

Kakwani defines the relative deprivation curve corresponding to the distribution x as the plot of $D_i^r(x)$ against the cumulative proportion of population $\frac{i}{n}(0 \leq i \leq n)$ and $D^r(x_0) = 1$. The relative deprivation curve is downward sloping but no definite conclusion can be drawn regarding its curvature (See Chakravarty et al., 1995).

Deprivation



The relative deprivation curve.

If the Lorenz curve coincides with the egalitarian line, then the relative deprivation curve coincides with the horizontal line OA. On the other hand, if there is maximum inequality, the curve coincides with CD.

Deprivation: BDP

Deprivation, in the whole society, is the sum of these sentiments:

The BDP aggregate measure of deprivation is a function $\mathbf{D}\colon \Omega \to \mathbb{R}_+$ such that:

$$\mathbf{D}(\pi, q) = \sum_{i=1}^{K} \pi_i \left(\sum_{k=1}^{i-1} \pi_k \right) \sum_{j=1}^{i-1} (\overline{q}_i - \overline{q}_j)\pi_j,$$

for all $(\pi, q) \in \Omega$.

What about time?

Does individual well-being depend on the individual's history?

Does it depend on other individuals' histories?

Deprivation: Bossert and D'Ambrosio (BD)

BD introduce a one-parameter class of dynamic individual deprivation measures.

BD modify Yitzhaki's index to take into account the part of deprivation generated by an agent's observation that others in it reference group move on to a higher level of income than himself.

The parameter reflects the relative weight given to these dynamic considerations, and the standard Yitzhaki index is obtained as a special case.

BD formalize an additional idea of Runciman that has not been explored in the literature yet:

“The more the people a man sees promoted when he is not promoted himself, the more people he may compare himself with in a situation where the comparison will make him feel relatively deprived” (Runciman, 1966, p.19).

Relative deprivation of an individual in BD framework is determined by the interaction of two components:

1. the average gap between the individual’s income and the incomes of all individuals richer than him (the traditional way of measuring individual deprivation);
2. a function of the number of people who were ranked below or equal in the previous-period distribution but are above the person under consideration in the current distribution.

BD use an axiomatic approach to derive classes of indices that capture these ideas.

Relative deprivation of an individual in BD framework is determined by the interaction of two components:

1. the average gap between the individual's income and the incomes of all individuals richer than him (the traditional way of measuring individual deprivation);
2. a function of the number of people who were ranked below or equal in the previous-period distribution but are above the person under consideration in the current distribution.

For all $(x^0, x^1) \in \mathbb{R}_+^{2n}$, where $\alpha \in [1, \infty)$ is a parameter,

$$D_i^\alpha(x^0, x^1) = \frac{\alpha^{|\text{number of people that passed}|}}{n} \sum_{j=i+1}^{n} (\overline{x}_j^1 - \overline{x}_i^1).$$

BD use ... ses of indices t...

Functioning failures

Well-being is multidimensional and income captures only a part of the story.

A simple way to proceed is to generate a distribution that counts the items individuals do not have.

Car/TV/Laptop: 0 for those with everything
1 for those missing one
2 for those missing two
3 for those missing everything

Functioning failures

We construct a deprivation score, q_i, for each population member, i, indicating the degree to which functionings that are considered relevant are not available to the agent.

Deprivation: BDP

Each individual feels alienated only in comparison with others with less functioning failures.

Bossert, D'Ambrosio & Peragine (BDP)

The members of the class of deprivation measures, $D_i\colon \Omega \to \mathbb{R}_+$, characterized by BDP are such that the degree of deprivation for a distribution (π, q) is obtained as the product of two terms with the following interpretation. The first factor is a multiple of the ratio of the number of agents who have fewer functioning failures than i and the population size. This number is interpreted as an inverse indicator of agent i's capacity to identify with other members of society–the lack of identification. The second factor is the average of the differences between q_i and the functioning failures of all agents having fewer functionings failure than i. This part captures the aggregate alienation experienced by i with respect to those who are better off. In particular the index is defined by:

$$D_i(\pi, q) = \left(\sum_{j=1}^{i-1} \pi_j\right) \sum_{j=1}^{i-1} (\overline{q}_i - \overline{q}_j)\pi_j,$$

for all $(\pi, q) \in \Omega$.

Bossert, D'Ambrosio & Peragine (BDP)

The members of the class of deprivation measures, $D_i\colon \Omega \to \mathbb{R}_+$, characterized by BDP are such that the degree of deprivation for a distribution (π, q) is obtained as the product of two terms with the following interpretation. The first factor is a multiple of the ratio of the number of agents who have fewer functioning failures than i and the population size. This number is interpreted as an inverse indicator of agent i's capacity to identify with other members of society–the lack of identification. The second factor is the average of the differences between q_i and the functioning failures of all agents having fewer functionings failure than i. This part captures the aggregate alienation experienced by i with respect to those who are better off. In particular the index is defined by:

$$D_i(\pi, q) = \left(\sum_{j=1}^{i-1} \pi_j\right) \sum_{j=1}^{i-1} (\overline{q}_i - \overline{q}_j)\pi_j,$$

for all $(\pi, q) \in \Omega$.

An application of deprivation, polarization, inequality

Deprivation in the São Paulo Districts:
Evidence from 2000
C. D'Ambrosio & R. Imanishi Rodrigues

This paper aims at capturing the **level of deprivation** of São Paulo's population in 2000 as suffered by its inhabitants in a non-income framework.

This paper aims at capturing the **level of deprivation** of São Paulo's population in 2000 as suffered by its inhabitants in a non-income framework.

We construct a measure of **functioning failure** which indicates the degree to which functionings that are considered relevant in the city districts are not available to the individuals.

This paper aims at capturing the **level of deprivation** of São Paulo's population in 2000 as suffered by its inhabitants in a non-income framework.

We construct a measure of **functioning failure** which indicates the degree to which functionings that are considered relevant in the city districts are not available to the individuals.

Deprivation is measured by **various indices** proposed in the literature: 1) the Yitzhaki, 2) the Esteban and Ray, and 3) the Bossert, D'Ambrosio and Peragine indices.

São Paulo represents a unique case study for deprivation.

The city is the **richest** city in Brasil in terms of GDP and shows striking disparities among its inhabitants (10.4 million in 2000) and worrisome indicators of economic well-being.

Many facts make of São Paulo a unique case study for deprivation: how does someone living in such a city relates to others?

One of the key variables in measuring deprivation is the **reference group**, that is the group with which a person compares itself.

We assume that in São Paulo the comparison takes place at the **district level**: individuals feel that they belong to the district where they live and derive within it their standards of comparison.

Since we believe that income is not always a good indicator of the command over resources nor of well-being of an individual, we follow the suggestion of Bossert, D'Ambrosio and Peragine (2005) and compute the indices on **deprivation scores based on various functionings**.

The aggregate indices

The BDP aggregate measure of deprivation is a function $\mathbf{D}\colon \Omega \to \mathbb{R}_+$ such that:

$$\mathbf{D}(\pi, q) = \sum_{i=1}^{K} \pi_i \left(\sum_{k=1}^{i-1} \pi_k \right) \sum_{j=1}^{i-1} (\overline{q}_i - \overline{q}_j)\pi_j, \quad (1)$$

for all $(\pi, q) \in \Omega$.

Similarly, aggregate deprivation suggested by Yitzhaki (1979), $\mathbf{I}\colon \Omega \to \mathbb{R}_+$, is given by:

$$\mathbf{I}(\pi, q) = \sum_{i=1}^{K} \pi_i \sum_{j=1}^{i-1} (\overline{q}_i - \overline{q}_j)\pi_j, \quad (2)$$

for all $(\pi, q) \in \Omega$, which is equal to the product of the mean of the vector q and the Gini coefficient, resulting is the absolute Gini coefficient. In the same way, the effective antagonisms felt by all members of the society is total polarization proposed by Esteban and Ray (1994), $\mathbf{P}\colon \Omega \to \mathbb{R}_+$, is defined by:

$$\mathbf{P}(\pi, q) = \sum_{i=1}^{K} \sum_{j=1}^{K} (\pi_i)^2 \, |q_i - q_j| \, \pi_j, \quad (3)$$

for all $(\pi, q) \in \Omega$.

Variables

From the microdata of the Censo 2000, we consider deprived an individual with the following characteristics:

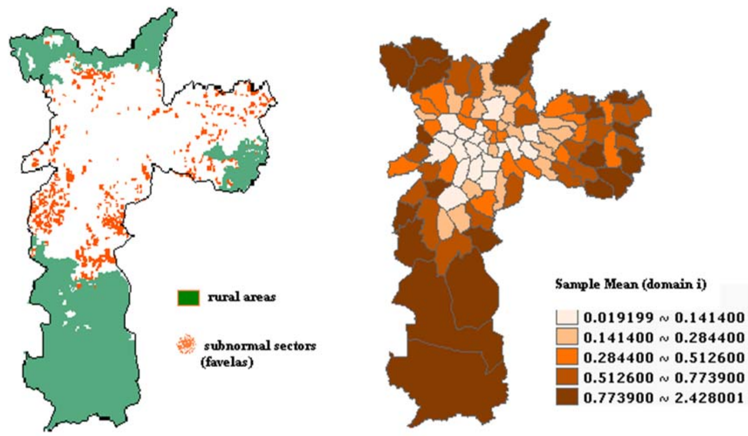
In domain **i)**:

1. **Lives in a rural area.**
2. **Lives in a favela.**
3. **Its dwelling is "improvised".**
4. **Its dwelling is of the one-room type.**
5. **Its dwelling is overcrowded.**
6. **Lives in a polluted area.**
7. **Lives in a place not served by good urban services.**

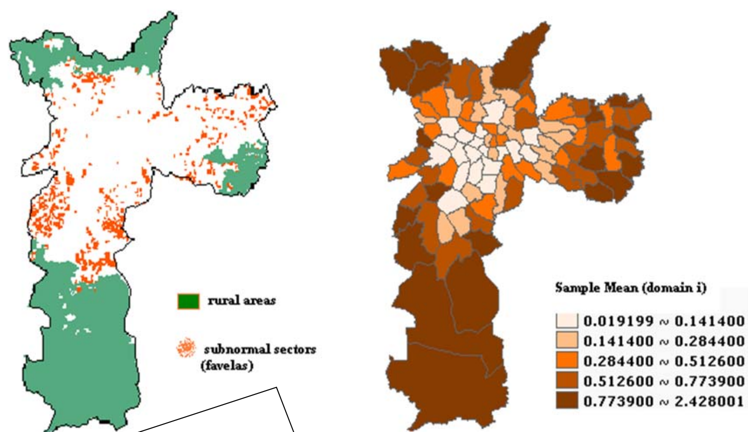
The individual functioning failure employed is the **number, unweighted, of the above listed variables** that the interviewed claimed to have, or not to have, depending on the variable.

Once we have obtained these scores for all individuals, we compute the **population shares** associated to the scores for **each district** separately.

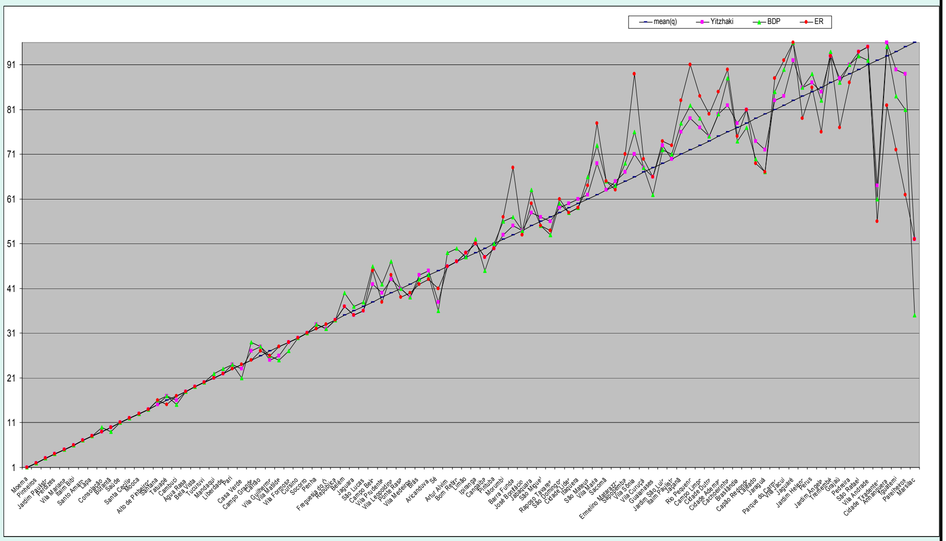
In the final step we proceede with the calculation of the indices.

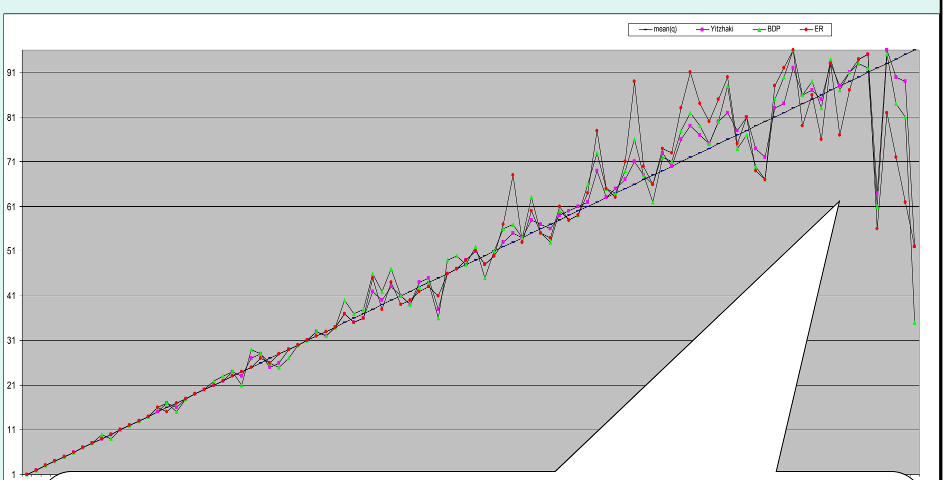


Source: Author´s elaboration/calculation from IBGE-Censo 2000.

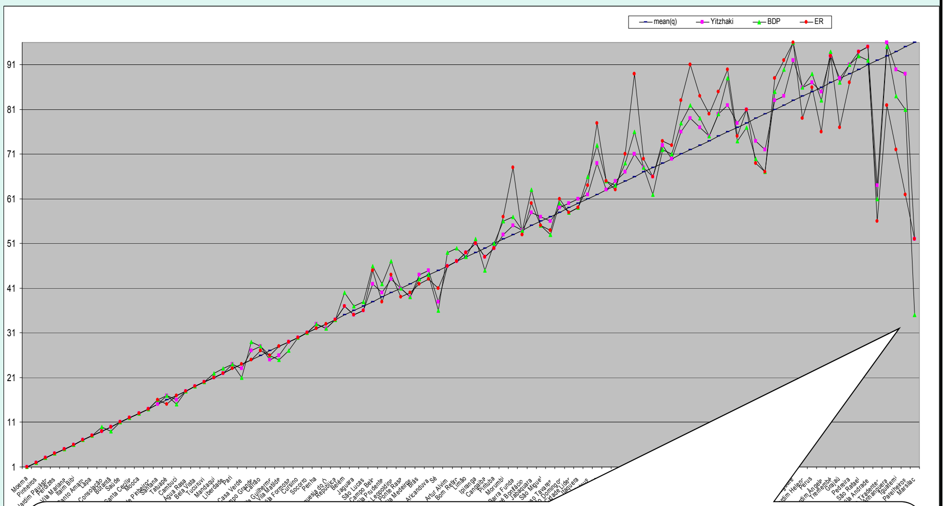


In **São Paulo precariousness is not exclusively a peripheral phenomenon**: some districts in the city center that are well served by urban services and have no favelas show means relatively high, owing to their proportions of other kinds of precarious housing units.

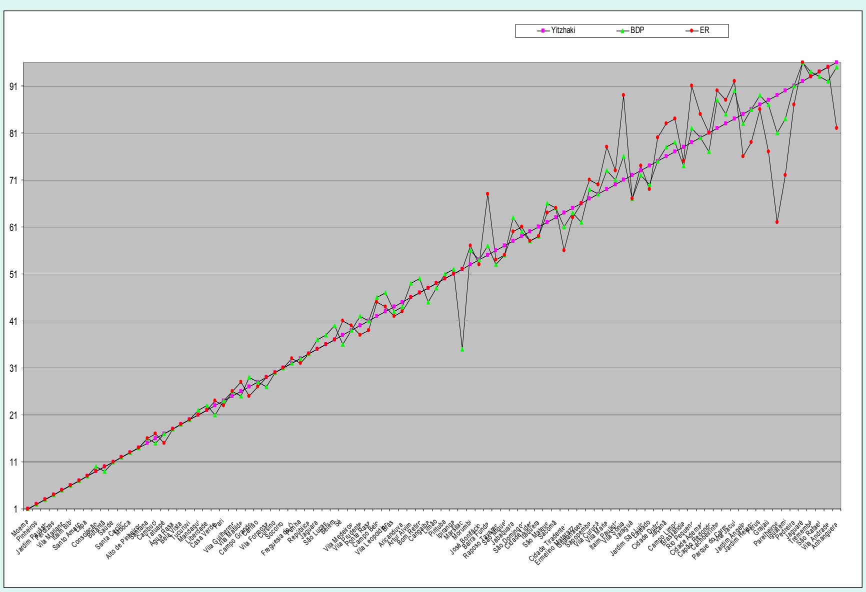


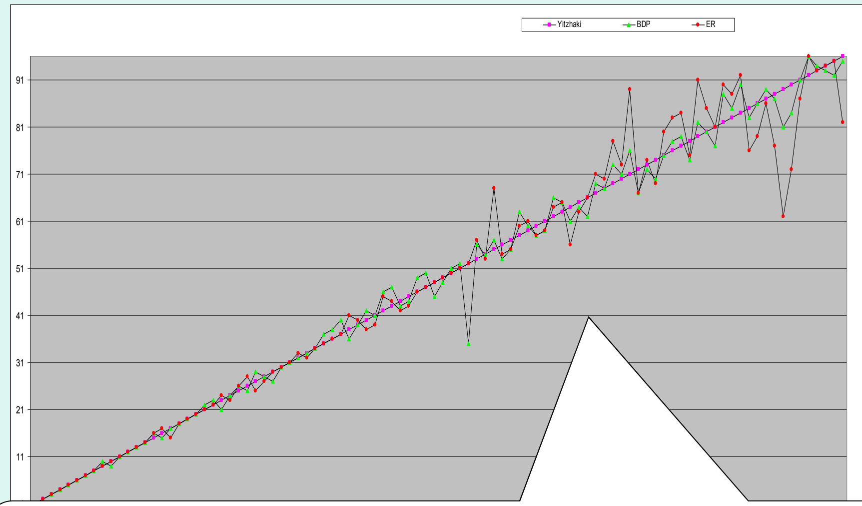


According to the deprivation indices, the **most** deprived districts based on the means of the functionings score **would be less deprived** than those occupying the middle positions. The three behavioral indices applied to the functionings of domain **i)** tend to reduce the importance of deprivation in districts with very high sample mean and very low population share having full access.

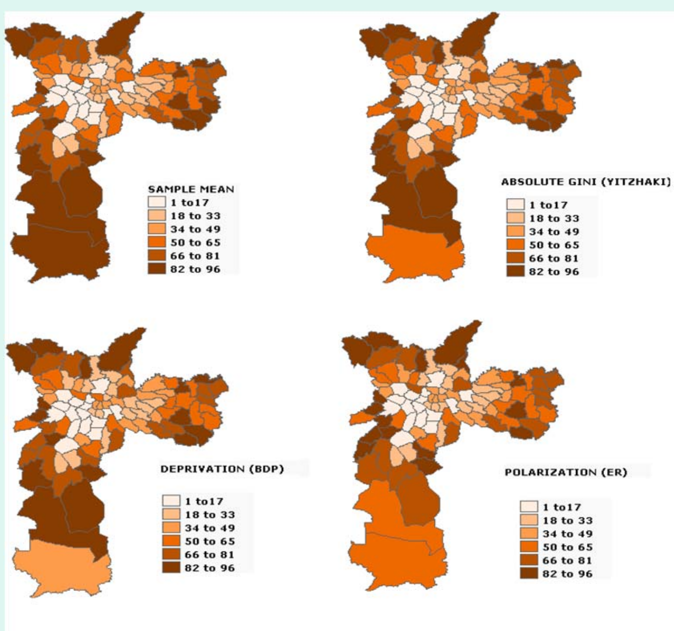


In the extreme south of the city, Marsilac (52), which is the worst in the ranking according to the sample mean, jumps to the middle of the orders of ER and down to the 35th position according to BDP; similarly, Parelheiros (55) and Cidade Tiradentes (25) fall considerably in position.

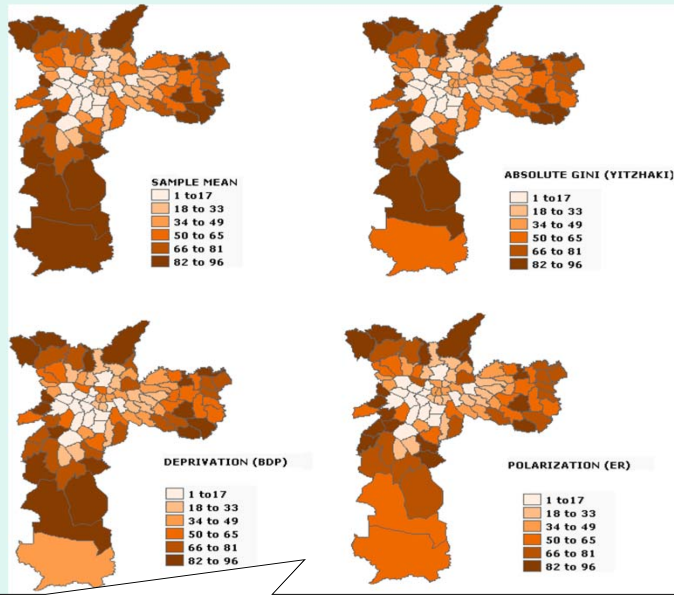




We confirm values being on average higher first and lower afterwards for BDP and ER, modifying Yitzhaki's rankings. **BDP and ER** register **higher** values in districts that are extremely **polarized** (high proportions of population with $q_j = 0$ and $q_j > 0$) and **lower** values in districts that are **homogeneously deprived** (very low population share with $q_j = 0$).



Source: Author's calculation from IBGE-Censo 2000.



The districts of the city center, where the majority of individuals have complete access, presents the lowest values according to all measures, for the others it depends on the index used.

As opposed to statistical measures such as sample means, the deprivation indices may **allow to better capture perceptions of individuals when comparing themselves to others**.

Thus **they better identify deprivation of poor individuals living in rich districts, and of poor individuals living in poor districts characterized by a homogeneous status of deprivation**.

Polarization and deprivation are important aspects of the Brazilian society, particularly so for cities like São Paulo where there is a considerable proportion of people “having” but the majority are “have-nots”.